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Transparency Property of One Dimensional Acoustic Wave Equations

by

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ABSTRACT

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This thesis proposes a new proof of the acoustic transparency theorem for material with a bounded variation. The theorem states that if the material properties (density, bulk modulus) is of bounded variation, the net power transmitted through the point $z = 0$ over a time interval $[-T, T]$ is greater than some constant times the energy at the time zero over a spatial interval $[0, Z]$, provided that T equals the time of travel of a wave from 0 to Z. This means the reflected energy of an input into the earth will be received. Otherwise, the reflections may not arrive at the surface. A proof gives a lower bound for material properties (density, bulk modulus) with bounded variation using sideways energy estimate. A different lower bound that works only for piecewise constant coefficients is also given. It gives a lower bound by analyzing reflections and transmissions of the waves at the jumps of the material properties. This thesis also gives an example to illustrate that the bounded variation assumption may not be necessary for the medium to be transparent. This thesis also discusses relations between the transparency property and the data of an inverse problem.

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Chapter 1

Introduction

In geophysics, people try to understand the structure of the earth from the input data and the output data of the seismic survey. The input data is generated by some devices and is applied to a surface of a material, for example the ground. Then a wave is formed and travels through the material. People usually record the data along the surface or slightly below the surface. The measured data is the acoustic pressure or surface motion resulting from the reflected or transmitted waves from sub-surfaces of the earth or materials.

This process can be well modeled by wave equations with non-smooth coefficients. As functions of positions in the earth, these coefficients describe the density, bulk modulus and wave speed in the earth. Since the earth is heterogeneous (see Figure 4 in Symes [15]), these functions are non-smooth.

This thesis chooses a one dimensional acoustic wave equation with a non-smooth coefficient, which could be used to describe many kinds of problems of the wave traveling through a plane layered medium. Plane layered medium is a good approximation of the earth because most mechanical properties of the earth changes largely in the vertical direction. Here the coefficient is a function of one variable, for example the depth of the earth. In a layered medium, reflections and transmissions will occur when waves travel from one layer to another (when meet a sub-surface). Constant density one dimensional acoustic wave equation (1.1) is a simplest model which captures the main features, such as a variable sound velocity c, transmitted and reflected waves.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} = 0, \tag{1.1}$$

Assume that the initial conditions are given

$$\begin{aligned} u(\cdot, t) &= u_0, \qquad t = 0 \\ \frac{\partial u}{\partial t}(\cdot, t) &= u_1, \quad t = 0 \end{aligned} \tag{1.2}$$

Define the energy of the initial data as

$$E(0) = \|[u_0, u_1]\|^2 = \frac{1}{2}\int_{\mathbb{R}} \left(\left(\frac{du_0}{dz}\right)^2 + \frac{u_1^2}{c^2} \right) dz$$

With appropriate assumption of the initial data and the function spaces of c and the solution u, one can define an operator W as

$$W[u_0, u_1] = \frac{\partial u}{\partial t}(0, \cdot). \tag{1.3}$$

I will show later that this operator W is well-defined.

The question we ask in this thesis is: if there is some energy deep in the earth, can it come to the surface of the earth at the expected time? In other words, is the operator W coercive? Is the operator $W^T W$ symmetric and positive definite? If the answer is yes, we say that the material has transparency property.

If the material is not transparent, arbitrarily small amount of energy could reach the surface. See Morlet [10] for an example that for highly oscillatory medium, the wave did not travel through it, which means the coercivity constant of W could be arbitrarily small. In this case, the change in wave field due to the change in the earth structure might not be evident at the surface. Thus there is not stable solution to the inverse problem associated with it. This study is a step towards the uniqueness of the solution to an inverse problem, which tries to find the structure of the medium (function c) from the measured *data* (measured $\frac{\partial u}{\partial t}(0,\cdot)$ with noise). The spectrum of $W^T W$ is the property of the input and output data (u_0, u_1 and *data*) of an inverse problem. I will discuss this in the Discussion chapter.

The transparency property says that the net power transmitted through the surface over a time interval is greater than some constant times the energy at the time zero over a corresponding spatial interval. The main result of this thesis is the following theorem.

Theorem 1.1 (Acoustic transparency theorem)

Suppose the initial data $u_0 \in H^1_{loc}(\mathbb{R})$ *and* $u_1 \in L^2_{loc}(\mathbb{R})$ *are supported on* $[0, Z]$ *and* $T = \int_0^Z \frac{1}{c}$. *Assume* $c \in BV(0, Z)$, $c(z) = c(0)$ *if* $z < \epsilon$ *and* $c(z) = c(Z)$ *if* $z > Z - \epsilon$ *with a small* $\epsilon > 0$, *i.e assume absorbing boundary conditions for domain* $[0, Z]$. *Also assume* $0 < c_{\min} \leq c \leq c_{\max}$. *Then*

$$k\|[u_0, u_1]\|^2 \leq \|W[u_0, u_1]\|^2 \leq K\|[u_0, u_1]\|^2. \tag{1.4}$$

with $K = c_{\max}$ *and*

$$k = \frac{c_{\min}^2}{2c_{\max}} \left(1 + Var(\mathrm{logc}) \exp(2\,Var(\mathrm{logc}))\right)^{-1}. \tag{1.5}$$

Here $BV(0, Z)$ stands for the bounded variation function defined on the interval $(0, Z)$ and $Var(\mathrm{logc})$ is the essential variation of logc. (See section 2.2 of this thesis for an accurate definition).

The acoustic transparency concept was introduced by Symes [13] in the 1980s. There were several related results and proofs since then. The description of the existing results will be given later in this chapter. As far as I know, the existing results are either only for a smaller subset of material than bounded variation function space or without the expression of k. This thesis constructs a new simple proof of the acoustic transparency theorem for the waves propagating through a medium with bounded variation.

Although bounded variation is a sufficient condition to make sure that the material is transparent, it may not be necessary. For wave equations with piecewise constant coefficients, an example is given to illustrate this observation, which comes from the analysis of transmissions and reflections of waves at the jumps of the coefficient. Please see the Discussion chapter for detailed analysis. This paper is also a step trying to find a function space of coefficients that is both necessary and sufficient for the acoustic transparency theorem.

Symes [13] introduced this theorem first without giving an explicit expression of k. It was stated with the assumption that the acoustic impedance is of bounded variation, where the impedance is defined on a time interval by changing the spatial variable with the time variable. Here impedance is $\frac{1}{c}$.

Symes [14] has a result for $c \in H^1$ that is close to this result. Lemma 1.4 of [14] says that the energy at the depth z over time interval $[z, 2T - z]$ (which is called sideways energy) is bounded above by the energy (L^2 norm) of the boundary data (on the domain of consideration) plus the energy of the initial data. The boundary has Neumann boundary conditions.

Another work that is closely related was done by Cox and Zuazua [3] 1995. They studied the 1D damped string problem. The second order equations are rewritten as a first order hyperbolic system $V' = AV$, where $V = [u, u']$ and

$$A = \begin{pmatrix} 0 & I \\ \frac{\partial^2}{\partial z^2} & -2a \end{pmatrix}.$$

They showed that the energy decay rate is the same as the supremum of the real part of the spectrum of the operator A. This is done by analyzing very carefully of the eigenvalues and eigenfunctions of the operator A. If we let $E(t)$ be the energy at time t, and let $\omega(a)$ be the energy decay rate with respect to the viscous damping $2a$, their result means that for all $\omega > \omega(a)$, there exists $C > 0$, such that $E(t) \leq CE(0)e^{2\omega t}$. This also gives us a lower bound of another kind of transparency property.

A similar result is also obtained in Demanet and Peyre [4] wherein they verify that the waveform inversion can be solved as a compressive sensing problem. The sparsity of the wave optimization problem gives that the L_1 energy of the wave field on a given time is bounded above by the energy of the initial data. The assumption Demanet and Peyre gave for this to be true is that the total variation of the impedance Var(logσ) < 1 or < 2 for periodic boundary conditions. Our theorem for the existence of an upper bound (L_2 norm of $\frac{\partial u}{\partial t}$) only needs Var(log($c$)) to be bounded, with $c = \frac{1}{\sigma}$. We follow their ideas of reflection, transmission and the discretization of the initial data with local support in obtaining a lower bound that only works for piecewise constant coefficient. This lower bound depends on $Var(\log c)$, but is different from the equation (1.5) in the acoustic transparency theorem.

In Chapter 2, I will give a clear description about the spaces of function u, c, initial condition u_0 and u_1. The well-posedness of the problem associated with the wave equation is also given in Chapter 2, following the structure of Chapter 2 of Stolk [12]. Then I will give you the domain and range of the operator W and show that it is well defined. Chapter 2 also includes the definition of bounded variation function. Bounded variation functions can be approximated by both piecewise constant functions (see DeVore [5] pp.65) and by smooth functions (see Evans and Gariepy [6] section 5.2). Other preliminary results, such as an energy estimate and the product of transmission coefficients, will also be given in this chapter.

Chapter 3 will give a complete proof of the acoustic transparency theorem. I also construct another approach to give a lower bound for material with properties of piecewise constant. This approach is created with the hope that it may be generalized to multi-dimensional problem.

In Chapter 4, I first build an example to show that the bounded variation assumption on the coefficient may not be necessary. Then I will set up an inverse problem associated with the wave equation (1.1) and also discuss the relations between the transparency theorem and the input and output data of an inverse problem.

Chapter 2

1D Acoustic Wave Equations and Preliminary Results

Chapter 1 states the wave equation (1.1) and gives the initial conditions (1.2) without giving specific information about the solution space and which kind of initial conditions should we assume. Thus this chapter will give a detailed information of u, c and initial condition u_0 and u_1, as well as the time and space domain of consideration. I will also show the well-posedness of the problem following the analysis of Chapter 2 of Stolk [12]. Then I will give a definition of operator W and show that it is well defined. Then I will define the energy norm and give an energy estimate.

The second part of this chapter will state some definitions and proof some preliminary results about wave equations with piecewise constant coefficients. One important result is the estimation of the product of transmission coefficient.

The definition of bounded variation function will be given in the third part of this chapter. Bounded variation functions can be approximated by both piecewise constant functions (see DeVore [5] pp.65) and by smooth functions (see Evans and Gariepy [6] section 5.2).

2.1 1D acoustic wave equations

One dimensional acoustic wave equations could be used to describe many kinds of problems arising from the wave traveling through a plane layered material. This paper deals mainly with second order acoustic wave equations with a constant density and a non-smooth sound velocity c, which describes the structure of a material. I state the wave equation (1.1) here.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} = 0, \tag{2.1}$$

In this equation, u is a function of $(z,t) \in \mathbb{R}^2$, where z is the depth variable and t is the time variable. The physical meaning of u is the acoustic pressure, which is the local deviation from the ambient pressure. c is the velocity of the wave traveling across the material. The velocity c is a function of $z \in \mathbb{R}$. Assume

$$0 < c_{\text{min}} \le c \le c_{\text{max}}.$$

Let M be a fixed large positive number. For a given $Z > 0$, and $T = \int_0^Z \frac{1}{c}$, we always choose $M > 2Z$. Thus by finite propagation speed of waves, the boundary values at $z = M$ and $z = -M$ could not influence the solution at domain $[0, Z]$ within time T.

Assume $c(z) = c(0)$ if $z < 0$ and $c(z) = c(Z)$ for $z > Z$. With this assumption, if there is a wave traveling out from the domain $[0, Z]$, it could not come back into this domain again within time T, since we choose $M > 2Z$. A slightly stronger assumption is posed: assume that $c(z) = c(0)$ for $z < \epsilon$ and $c(z) = c(Z)$ for $z > Z - \epsilon$ with a small $\epsilon > 0$ for the rest of this thesis.

Assume

$$u \in L^2(\mathbb{R}, H_0^1[-M, M]) \cap H^1(\mathbb{R}, L^2[-M, M]).$$

Let $u(t) = u(\cdot, t)$ be a function of z for fixed t. The notation of spaces means that $u(t)$ is an L^2 function of t with values in $H_0^1[-M, M]$ and $u'(t) = \dfrac{\partial u(\cdot, t)}{\partial t}$ is an L^2 function of t with values in $L^2[-M, M]$. Initial conditions given in Chapter 1 are

$$\begin{aligned} u(\cdot, t) &= u_0, \quad & t = 0 \\ \frac{\partial u}{\partial t}(\cdot, t) &= u_1, \quad & t = 0 \end{aligned} \tag{2.2}$$

with $u_0 \in H_0^1[-M, M]$ and $u_1 \in L^2[-M, M]$.

I follow chapter 2 of Stolk [12] to show that the initial value problem (1.1),(1.2) is well-posed. When the coefficient c is not differentiable, wave equation (1.1) is understood in the sense of distribution. By integrating with a test function $v \in C_0^\infty((-M, M) \times \mathbb{R})$, the wave equation (1.1) means

$$\int_{\mathbb{R}^2} \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} - \frac{1}{c^2} \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \right) dx dt = 0, \tag{2.3}$$

for all $v \in C_0^\infty((-M, M) \times \mathbb{R})$. This formula is called the **weak form of wave equation** (1.1).

Although the density of the medium ρ is constant, there are reflections if the bulk modulus $\kappa = \rho c^2$ has jumps. By assuming the constant density, we do not lose any generality. In fact, the general acoustic wave equation is

$$\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left(\frac{1}{\rho} \frac{\partial u}{\partial x} \right) = 0,$$

where the ρ and c are functions of x. It has a weak form

$$\int_{\mathbb{R}^2} \left(\frac{1}{\rho} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} \right) dxdt = 0$$

with $v \in C_0^\infty((-M, M) \times \mathbb{R})$. It is natural to assume $0 < \rho_{\min} \leq \rho \leq \rho_{\max}$. If we perform a change of variable by $\frac{dz}{dx} = \frac{1}{\rho(x)}$, $z(0) = 0$, we get exactly the weak form of wave equation (2.3).

Assume $u, w \in L^2(\mathbb{R}, H_0^1[-M, M]) \cap H^1(\mathbb{R}, L^2[-M, M])$. They could be solutions of the wave equation (2.3) with whatever initial conditions and boundary conditions they have.

Define the energy inner product, the acoustic energy and the energy norm at time t on an interval $[0, Z]$ as

$$\langle [u(t), u'(t)], [w(t), w'(t)] \rangle = \frac{1}{2} \int_0^Z \left(\frac{\partial u}{\partial z} \frac{\partial w}{\partial z} + \frac{1}{c^2} \frac{\partial u}{\partial t} \frac{\partial w}{\partial t} \right) dz.$$

$$E(t) = \langle [u(t), u'(t)], [u(t), u'(t)] \rangle. \tag{2.4}$$

$$\|[u(t), u'(t)]\| = \sqrt{E(t)}.$$

2.1.1 Initial value problem (1.1), (1.2) is well-posed

Define two bi-linear forms according to the weak form (2.3)

$$\begin{aligned} a(u, v) &= \int_R \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} dz \quad \text{for } u, v \in H_0^1[-M, M], \\ b(u, v) &= \int_R \frac{1}{c^2} uvdz \quad \text{for } u, v \in L^2[-M, M]. \end{aligned} \tag{2.5}$$

Applying Poincare inequality to $u \in H_0^1[-M,M]$ gives that $\|u\|_{L^2[-M,M]} \leq C\|u'\|_{L^2[-M,M]}$ with C depends only on M. For supp$u \subset [0,1]$, $C = \dfrac{1}{\pi}$. Then we have

$$a(u,u) \geq \alpha \|u\|^2_{H^1[-M,M]},$$

with $\alpha = \dfrac{1}{1+C^2}$, where C depends only on the support of u. It is clear that

$$b(u,u) \geq \frac{1}{c^2_{\max}} \|u\|^2_{L^2[-M,M]}.$$

$L^2[-M,M]$ and $H_0^1[-M,M]$ are Hilbert spaces. Following Riesz representation theorem, there exists an operator $A : H_0^1[-M,M] \to H^{-1}[-M,M]$, such that

$$a(u,v) = \langle Au, v\rangle = \langle u, Av\rangle.$$

Similarly, since $L^2[-M,M]$ is self-dual, there exists an operator $B : L^2[-M,M] \to L^2[-M,M]$, such that

$$b(u,v) = \langle Bu, v\rangle = \langle u, Bv\rangle.$$

Since $u \in L^2(\mathbb{R}, H_0^1[-M,M]) \cap H^1(\mathbb{R}, L^2[-M,M])$, it means $u(t)$ is an $L^2(\mathbb{R})$ function and $u'(t) \in L^2(\mathbb{R}, L^2[-M,M])$ is an $L^2(\mathbb{R})$ function. However, since $u \in L^2(\mathbb{R}, H_0^1[-M,M])$ we cannot take the trace $u(t)$ and say that $u(t) \in H_0^1[-M,M]$ for fixed t.

By theorem 4.5.12 of Hormander [8], Section 4.5, $u(t)$ is Holder continuous of order $\dfrac{1}{2}$ with values in $L^2[-M,M]$, i.e. $u \in C^{\frac{1}{2}}(\mathbb{R}, L^2[-M,M])$. It means for a given t, $u(t) \in L^2[-M,M]$.

By the variational formulation (2.5) and the operators A and B, $u'' = -Au \in L^2(\mathbb{R}, H^{-1}[-M,M])$. Thus by the same theorem of Hormander [8], $u'(t)$ is a Holder continuous function of order $\frac{1}{2}$ with values in $H^{-1}[-M, M]$, i.e. $u \in C^{\frac{1}{2}}(\mathbb{R}, H^{-1}[-M, M])$. It means for a given t, $u'(t) \in H^{-1}[-M, M]$.

It is also known that $H_0^1[-M, M] \subset L^2[-M, M] \subset H^{-1}[-M, M]$. Thus initial value problem (1.1), (1.2) with $u_0 \in H_0^1[-M, M]$ and $u_1 \in L^2[-M, M]$ are well-posed.

Theorem 2.4.5 of Stolk [12] showed that the wave equations with a right hand side $f \in L^2(\mathbb{R}, L^2[-M, M])$, an L^∞ coefficient and the same initial data (1.2) has a solution $u \in C(\mathbb{R}, H_0^1[-M, M]) \cap C^1(\mathbb{R}, L^2[-M, M])$, which is unique in $L^2(\mathbb{R}, H_0^1[-M, M]) \cap H^1(\mathbb{R}, L^2[-M, M])$.

Theorem 2.8.2 of Stolk [12] gave a continuous dependence of the solution of wave equations on coefficients.

Assume V and H are two Hilbert spaces. In this thesis, $V = H_0^1[-M, M]$ and $H = L^2[-M, M]$. V' is the dual space of V and H is self-dual space. Two operators $A : V \to V'$ and $B : H \to H$ are defined as before.

Lemma 2.1 (Theorem 2.8.2 of Stolk [12])

Assume operators $A_n : V \to V'$ and $B_n : H \to H$ satisfy $\|(A_n - A)v\|_{V'} \to 0$ for $\forall v \in V$ and $\|(B_n - B)u\|_H \to 0$ for $\forall u \in H$. If V is compactly embedded in H and u_n solves the weak form of wave equations associated with A_n and B_n (equation (2.3) with coefficient c_n), then

$$u_n \to u$$

in $C([0, T], V) \cap C^1([0, T], H)$.

2.1.2 Operator W is well defined

The following lemma shows that traces of $\dfrac{\partial u}{\partial t}$ on $z = 0$ and $z = Z$ are well defined.

Lemma 2.2 *Assume the initial conditions $u_0 \in H_0^1[-M, M]$ and $u_1 \in L^2[-M, M]$ are supported on $[0, Z]$ and $c \in L^\infty(0, Z)$. Assume that u solves the wave equation (1.1) with initial conditions u_0, u_1 and $c(z) = c(0)$ if $z < \epsilon$ and $c(z) = c(Z)$ if $z > Z - \epsilon$ with $\epsilon > 0$. Then traces of $\dfrac{\partial u}{\partial t}$ on $z = 0$ and $z = Z$ are well defined and*

$$\frac{\partial u}{\partial t}(0, \cdot),\ \frac{\partial u}{\partial t}(Z, \cdot) \in L^2[-T, T].$$

for any $T > 0$.

Assume the initial conditions u_0 and u_1 are supported on $[0, Z]$. Since $BV(0, Z) \subset L^\infty(0, Z)$, with this lemma, define operator $W : H_0^1[0, Z] \times L^2[0, Z] \to L^2[-T, T]$ as $W[u_0, u_1] = \dfrac{\partial u}{\partial t}(0, \cdot)$ for any time $T > 0$. Here u is a solution to the wave equation (1.1) with initial conditions u_0 and u_1.

Now let's prove Lemma 2.2.

Proof. Since $u_0 \in H_0^1[-M, M]$ and $u_1 \in L^2[-M, M]$ are supported on $[0, Z]$, our problem satisfies the assumption of the Theorem 2.4.5 of Stolk [12]. Thus choose $M > 2Z$ and we have that $u \in L^2(\mathbb{R}, H_0^1[-M, M]) \cap H^1(\mathbb{R}, L^2[-M, M])$ and $u \in C(\mathbb{R}, H_0^1[-M, M]) \cap C^1(\mathbb{R}, L^2[-M, M])$ outside a set of measure 0. Then $\dfrac{\partial u}{\partial z} \in L^2(\mathbb{R}, L^2[-M, M])$ and $\dfrac{\partial u}{\partial z} \in C(\mathbb{R}, L^2[-M, M])$ outside a set of measure 0.

According to the property of c we assumed, $c(z) = c(0)$ is a constant for $z \le \epsilon$. Let $\eta \in C^\infty(\mathbb{R})$ with $\eta(z) = 1$ if $z \le 0$ and $\eta(z) = 0$ if $z \ge \epsilon > 0$. $w = \eta u$ is a weak solution of the following initial value problem on $\mathbb{R} \times \mathbb{R}$

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = 2\frac{d\eta}{dz}\frac{\partial u}{\partial z} + \frac{d^2\eta}{dz^2}u \tag{2.6}$$

with zero initial data and $c = c(0)$ for all $z \in \mathbb{R}$.

Denote the right hand side of the above wave equation by f.

The Green's function for this wave equation with constant c is

$$G(z,t) = \frac{1}{2c}H(ct - |z|),$$

where H is the Heaviside step function.

Then the solution to the wave equation can be written as

$$w(z,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} G(z-y, t-\tau)f(y,\tau)d\tau dy = \frac{1}{2c}\int_{\mathbb{R}}\int_{\mathbb{R}} H(c(t-\tau)-|z-y|)f(y,\tau)d\tau dy.$$

Thus we have

$$w(z,t) = \frac{1}{2c}\int_{\mathbb{R}}\int_{-\infty}^{t-\frac{|z-y|}{c}} f(y,\tau)d\tau dy$$

and

$$\frac{\partial w}{\partial t}(z,t) = \frac{1}{2c}\int_{\mathbb{R}} f\left(y, t - \frac{|z-y|}{c}\right)dy.$$

The support of right hand side $f = 2\frac{d\eta}{dz}\frac{\partial u}{\partial z} + \frac{d^2\eta}{dz^2}u$ is $[0,\epsilon] \times \mathbb{R}$. Since $u_0 \in H_0^1[-M,M]$, $u_1 \in L^2[-M,M]$ are supported on $[0,Z]$ and the coefficient of the wave equation (1.1) is of bounded variation, which is a subset of L^∞, by Theorem 2.4.5 of Stolk [12], our initial value problem (1.1), (1.2) has a unique solution $u \in L^2(\mathbb{R}, H_0^1[-M,M]) \cap H^1(\mathbb{R}, L^2[-M,M])$. By the proof of Theorem 2.4.5 of Stolk [12] and Poincare inequality, this gives

$$\|u\|^2_{H^1([-M,M]\times\mathbb{R})} \le C(\|u_0\|^2_{H_0^1[-M,M]} + \|u_1\|^2_{L^2[-M,M]}),$$

for some constant C. Thus we have

$$\int_0^\epsilon \int_{\mathbb{R}} f^2 dt dz = \int_0^\epsilon \int_{\mathbb{R}} (2\frac{d\eta}{dz}\frac{\partial u}{\partial z}+\frac{d^2\eta}{dz^2}u)^2 dt dz \leq \hat{C} \left\| u \right\|^2_{H^1([0,\epsilon]\times\mathbb{R})} \leq \hat{C} \left\| u \right\|^2_{H^1([-M,M]\times\mathbb{R})},$$

which gives that $f \in L^2([0,\epsilon]\times\mathbb{R})$. Since $u_0 \in H_0^1[-M,M]$, $u_1 \in L^2[-M,M]$ are supported on $[0,Z]$, which implies that $u_0 \in H_0^1[0,Z]$ and $u_1 \in L^2[0,Z]$, the above argument shows that the map $[u_0,u_1] \mapsto f$ from $H_0^1[0,Z]\times L^2[0,Z]$ to $L^2([0,\epsilon]\times\mathbb{R})$ is bounded in L^2 norm.

$$\frac{\partial w}{\partial t}(z,t) = \frac{1}{2c}\int_0^\epsilon f\left(y, t-\frac{|z-y|}{c}\right)dy$$

Assume f is continuous, then we have $\frac{\partial w}{\partial t}$ is continuous on $\mathbb{R}^2$ and $\frac{\partial w}{\partial t}(0,\cdot)$ is well-defined. Use Cauchy-Schwarz inequality and get

$$\begin{aligned} & \int_0^T \left(\frac{\partial w}{\partial t}(0,\cdot)\right)^2 dt \\ = \;& \frac{1}{4c^2}\int_0^T\left(\int_0^\epsilon f\left(y,t-\frac{|y|}{c}\right)dy\right)^2 dt \\ \leq \;& \frac{\epsilon^2}{2c^2}\left(\int_0^T\int_0^\epsilon f\left(y,t-\frac{y}{c}\right)^2 dy dt\right) \end{aligned}$$

Thus if f is continuous on $[0,\epsilon]\times\mathbb{R}$, $\frac{\partial w}{\partial t}(0,\cdot)$ is continuous on $[0,T]$ and $\left\|\frac{\partial w}{\partial t}(0,\cdot)\right\|^2_{L^2([0,T])} \leq \frac{\epsilon^2}{2c^2}\left\|f\right\|^2_{L^2([0,\epsilon]\times\mathbb{R})}$. Thus the map $f \mapsto \frac{\partial w}{\partial t}(0,\cdot)$ from $C^0([0,\epsilon]\times\mathbb{R})$ to $C^0[0,T]$ is bounded in the L^2 norm.

For $f \in L^2([0,\epsilon]\times\mathbb{R})$, since continuous function space is dense in L^2 space, we can find a sequence of continuous function f_j that converges to f in L^2. Define $\frac{\partial w_j}{\partial t}(0,\cdot)$ for each f_j. It is easy to see that $\frac{\partial w_j}{\partial t}(0,\cdot)$ is a Cauchy sequence in $L^2([0,T])$ and hence converges to a function in $L^2([0,T])$, which we define as $\frac{\partial w}{\partial t}(0,\cdot)$.

Since $\left\|\frac{\partial w_j}{\partial t}(0,\cdot)\right\|^2_{L^2([0,T])} \leq \frac{\epsilon^2}{2c^2}\|f_j\|^2_{L^2([0,\epsilon]\times\mathbb{R})}$ for all j, by passing to the limit, we have $\left\|\frac{\partial w}{\partial t}(0,\cdot)\right\|^2_{L^2([0,T])} \leq \frac{\epsilon^2}{2c^2}\|f\|^2_{L^2([0,\epsilon]\times\mathbb{R})}$ for $f \in L^2([0,\epsilon]\times\mathbb{R})$. Thus the map $f \mapsto \frac{\partial w}{\partial t}(0,\cdot)$ from $L^2([0,\epsilon]\times\mathbb{R})$ to $L^2([0,T])$ is bounded.

The trace of w on $z = 0$ is therefore well-defined and of class H^1.

By Theorem 2.4.5 of Stolk [12], the above zero initial and boundary value problem with inhomogeneous right hand side has a unique weak solution, i.e. $w = \eta u$. Thus $w(0,\cdot) = u(0,\cdot)$ and $\frac{\partial w}{\partial t}(0,\cdot) = \frac{\partial u}{\partial t}(0,\cdot)$.

Thus we have $\frac{\partial u}{\partial t}(0,\cdot) \in L^2[0,T]$.

By the composition of the two maps $[u_0,u_1] \mapsto f$ and $f \mapsto \frac{\partial u}{\partial t}(0,\cdot)$, we have that if $[u_0,u_1] \in H^1_0[0,Z] \times L^2[0,Z]$ and u solves the weak form of wave equation (2.3) with $[u_0,u_1]$ as the initial conditions and Dirichlet boundary conditions on domain $[-M,M]$, then $\frac{\partial u}{\partial t}(0,\cdot) \in L^2[0,T]$.

Similarly, $\frac{\partial u}{\partial t}(0,\cdot) \in L^2[-T,0]$ and $\frac{\partial u}{\partial t}(Z,\cdot) \in L^2[-T,T]$.

Thus the operator $W : H^1_0[0,Z] \times L^2[0,Z] \to L^2[-T,T]$ defined by equation (1.3) is well defined.

2.1.3 Energy estimate

This subsection will give an energy estimate of the energy $E(t)$ defined as equation (2.4). I will use it to obtain the upper bound of the acoustic transparency theorem in chapter 3.

The solution to the wave equation can be approximated by the convolution of the solution with a smooth function in time. The result of the convolution also gives a solution to the wave equation which is smooth in time. Then it is easy to show that the smooth solution satisfies the energy estimate by differentiation and integration by parts.

From lemma 2.2,

$$\frac{\partial u}{\partial t}(0,\cdot),\ \frac{\partial u}{\partial t}(Z,\cdot) \in L^2[-T,T].$$

Thus the equation (2.8) is well defined.

Lemma 2.3 *With the same assumption of the acoustic transparency theorem, i.e.* $u \in L^2(\mathbb{R}, H_0^1[-M,M]) \cap H^1(\mathbb{R}, L^2[-M,M])$ *solves the wave equation (1.1) with absorbing boundary conditions. The energy which defined by expression (2.4)*

$$E(t) = \frac{1}{2}\int_0^Z \left(\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2\right) dz. \tag{2.7}$$

satisfies

$$E(t) = E(0) - \frac{1}{c(0)}\int_0^t \left(\frac{\partial u}{\partial t}(0,s)\right)^2 ds - \frac{1}{c(Z)}\int_0^t \left(\frac{\partial u}{\partial t}(Z,s)\right)^2 ds \tag{2.8}$$

Proof. This proof is following the idea of smoothing, see the proof of Lemma 2.4.1 of Stolk [12].

Define $\eta_m \in C_0^\infty(\mathbb{R})$, $m = 1, 2, \cdots$ as

$$\eta_m(t) \geq 0$$
$$\int_{\mathbb{R}} \eta_m(t)dt = 1$$
$$\text{supp}\eta_m(t) = \left[-\frac{1}{m}, \frac{1}{m}\right]$$

Define the convolution of u with η_m

$$u_m(z,t) = (\eta_m * u)(z,t) = \int_{\mathbb{R}} \eta_m(t-s)u(z,s)ds.$$

By the commutativity of convolution, $\dfrac{\partial^2 u_m}{\partial t^2} = \dfrac{\partial^2 \eta_m}{\partial t^2} * u = \eta_m * \dfrac{\partial^2 u}{\partial t^2}$. And also $\dfrac{\partial^2 u_m}{\partial z^2} = \eta_m * \dfrac{\partial^2 u}{\partial z^2}$. Thus u_m satisfies the wave equation (1.1). Also notice that for any $n \in \mathbb{N}$, $\dfrac{\partial^n u_m}{\partial t^n} = \dfrac{\partial^n \eta_m}{\partial t^n} * u$. Thus $\dfrac{\partial^n u_m}{\partial t^n}$ satisfies the wave equation (1.1) as well.

Let $E_m(t)$ denote the corresponding energy of u_m. Differentiate $E_m(t)$ with respect to t,

$$\begin{aligned}
\frac{dE_m(t)}{dt} &= \int_0^Z \left(\frac{\partial u_m}{\partial z}\frac{\partial^2 u_m}{\partial z \partial t} + \frac{1}{c^2}\frac{\partial u_m}{\partial t}\frac{\partial^2 u_m}{\partial t^2} \right) dz \\
&= \int_0^Z \left(\frac{\partial u_m}{\partial z}\frac{\partial^2 u_m}{\partial z \partial t} + \frac{\partial u_m}{\partial t}\frac{\partial^2 u_m}{\partial z^2} \right) dz \\
&= \frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(Z,t) - \frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(0,t)
\end{aligned}$$

and

$$E_m(t) = E_m(0) + \int_0^t \left(\frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(Z,t) - \frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(0,t) \right) dt \tag{2.9}$$

$$u_m(z,t) = -\int_{t-\frac{1}{m}}^{t+\frac{1}{m}} \eta_m(t-s)u(z,s)ds$$

By finite propagation speed property of waves, and $u(z,0) = 0$ for $z < 0$, we get $u_m(z,0) = 0$ for $z \le -\dfrac{1}{m}$.

Since $u_0(z) = u_1(z) = 0$ for $z < 0$, we get that $u_m(0,z) = \dfrac{\partial u_m}{\partial t}(0,z) = 0$ for $z \le -\dfrac{1}{m}$.

For $z \le 0$ and $t > \dfrac{1}{c(0)m}$, by finite propagation speed of waves, u_m has the following expression

$$u_m(z,t) = f(z + c(0)t), \qquad \text{for } z \le 0,\ t > \frac{1}{c(0)m},$$

where the support of $f \subset \mathbb{R}^+$.

Then for $z < 0$ and $t > \dfrac{1}{c(0)m}$, $\dfrac{\partial u_m}{\partial z} = \dfrac{1}{c(0)}\dfrac{\partial u_m}{\partial t}$.

Similarly, for $z > Z$ and $t > \dfrac{1}{c(0)m}$, the solution of wave equation has the form

$$u_m(z,t) = f(z - c(Z)t)$$

thus satisfies $\dfrac{\partial u_m}{\partial z} = -\dfrac{1}{c(Z)}\dfrac{\partial u_m}{\partial t}$.

Then equation (2.9) becomes

$$\begin{aligned} E_m(t) = \quad & E_m(0) - \int_{\frac{1}{c(0)m}}^{t} \left(\frac{1}{c(0)} \left(\frac{\partial u_m}{\partial t}(0,t) \right)^2 + \frac{1}{c(Z)} \left(\frac{\partial u_m}{\partial t}(Z,t) \right)^2 \right) dt \\ & - \int_0^{\frac{1}{c(0)m}} \left(\frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(Z,t) - \frac{\partial u_m}{\partial z}\frac{\partial u_m}{\partial t}(0,t) \right) dt \end{aligned} \tag{2.10}$$

As we discussed in Chapter 1, there is a solution $u \in C(\mathbb{R}, H_0^1[0,Z]) \cap C^1(\mathbb{R}, L^2[0,Z])$, which is unique in $L^2(\mathbb{R}, H_0^1[0,Z]) \cap H^1(\mathbb{R}, L^2[0,Z])$. Thus by Young's inequality

$$\begin{aligned} & \int_0^Z \left(\frac{\partial u_m}{\partial t} - \frac{\partial u}{\partial t} \right)^2 dz \\ = \; & \int_0^Z \left(\int_{-\frac{1}{m}}^{\frac{1}{m}} \eta_m(s) \left(\frac{\partial u}{\partial t}(z,t-s) - \frac{\partial u}{\partial t}(z,t) \right) ds \right)^2 dz \\ \leq \; & \int_0^Z \int_{-\frac{1}{m}}^{\frac{1}{m}} \left(\frac{\partial u}{\partial t}(z,t-s) - \frac{\partial u}{\partial t}(z,t) \right)^2 ds dz, \end{aligned}$$

which goes to 0 as $m \to 0$. Similarly $\dfrac{\partial u_m}{\partial z} \to \dfrac{\partial u}{\partial z}$ in $L^2[0,Z]$ for any t.

Thus let $m \to 0$, $E_m(t) \to E(t)$. If let $m \to 0$ in equation (2.10), the last term goes to 0 and the rest becomes the energy identity we want.

2.2 Bounded variation functions

Since the main result is based on the space of functions of bounded variation, let me introduce the definition of this space, which need the definition of point of approximate continuity of a function (see Evans and Gariepy [6] pp.47).

Definition 2.1 *A point z is a* **point of approximate continuity** *if and only if*

$$c(z) = \operatorname{ap}\lim_{x \to z} c(x),$$

where $a = \operatorname{ap}\lim_{x \to z} c(x)$ if and only if for every $\epsilon > 0$,

$$\lim_{r \to 0} \frac{\mathcal{L}^1(\{|x-z| < r\} \cap \{|c-a| > \epsilon\})}{\mathcal{L}^1(\{|x-z| < r\})} = 0.$$

Here $\mathcal{L}^1$ is one dimensional Lebesgue measure.

Remark 2.1 *A point of continuity is a point of approximate continuity.*

Assume a function f is continuous at a point z. That is for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $|x - z| < \delta$, $|f(x) - f(z)| < \epsilon$. Thus for $r < \delta$,

$$\{x : |f(x) - f(z)| > \epsilon\} \cap \{x : |x - z| < r\} = \emptyset,$$

which has Lebesgue measure zero. Thus, z is also a point of approximate continuity.

The converse statement is false. For

$$f(x) = \begin{cases} 1, & x \text{ irrational or } x = 0 \\ 0, & x \text{ rational and } x \neq 0. \end{cases}$$

Since rational numbers intersecting with $\{x : |x - 0| < r\}$ for any $r > 0$ has Lebesgue measure zero, $f(x)$ is approximately continuous at 0. However, it is obvious that $f(x)$ is not continuous at any point.

Remark 2.2 *Step functions are not approximately continuous at discontinuous points.*

The essential variation of a function can be described as follows (see Evans and Gariepy [6] pp.216).

Definition 2.2 *Let c be a Lebesgue measurable function defined on an interval (a, b). The* **essential variation** *of c on this interval is*

$$\mathrm{Var}(c) = \sup_{\mathcal{P}} \sum_{i=1}^{N} |c(z_i) - c(z_{i-1})| \, .$$

where $\mathcal{P}$ is the set of all the finite partitions of the interval (a, b). N depends on the partition. Each z_i of the partition is a point of approximate continuity of c.

If the essential variation is finite, we call c a function of **bounded variation** and denote by $BV(a, b)$ all the functions with bounded variation on an interval (a, b).

The bounded variation functions can be approximated by smooth functions. Since I need this smooth approximation, I state the result here.

Lemma 2.4 (Smooth approximation)

If $c \in BV(0, Z)$, there exist a sequence of functions $\{c_k\}_{k=1}^{+\infty} \subset BV(0, Z) \cap C^{\infty}(0, Z)$ such that $c_k \to c$ in $L^1(0, Z)$ and $\mathrm{Var}(c_k) \to \mathrm{Var}(c)$ as $k \to +\infty$.

Please refer to section 5.2 of Evans and Gariepy [6] for a proof. I also include this proof in the Appendix A.2.

2.3 Preliminary results for wave equations with piecewise constant coefficients

Preliminary results which can make the proofs easier to read are given in this section. These results are also interesting in themselves.

Let me describe my idea of the proof first in words. The acoustic transparency theorem gives the estimates of the ratio of the energy of the trace on $z = 0$ over a time interval and energy over a spatial interval at time $t = 0$. Let's call K the upper bound and k the lower bound. The upper bound K is obtained by manipulating the energy $E(t)$ defined as equation (2.4), i.e. by differentiating and integration by parts. Comparing with the proof of upper bound, the approaches used in this thesis to deal with the lower bound are more interesting.

This thesis gives the lower bound for bounded variation coefficients using the sideways energy (see Symes [14]), which depends on the travel time function $\tau(z) = \int_0^z \frac{1}{c} dz$ and its inverse function. I also give a lower bound for wave equations with piecewise constant coefficients by analyzing the reflections and transmissions of the waves at the jumps of the sound velocity. This result does not depend on the travel time function. It gives a hint of a proof for multi-dimensional problems.

Assume that u solves the wave equation (1.1) with a piecewise constant c. Here u is defined over $\mathbb{R}$. If consider finite spatial interval $[0, Z]$, I assume $c(z) = c(0)$ for $z \leq \epsilon$ and $c(z) = c(Z)$ for $z \geq Z - \epsilon$ for a small $\epsilon > 0$.

Let $\mathcal{P} = \{z_0 = 0, z_1, \cdots, z_N = Z\}$ be a partition of $[0, Z]$. Assume

$$c(z) = c_i \text{ if } z \in (z_{i-1}, z_i).$$

And c also satisfies $0 < c_{\min} \leq c \leq c_{\max}$.

In this section, I will give a description of the up-coming and down-going waves first. And then define the reflections and transmissions of waves at the jumps of coefficients. I will also define the purely transmitted part of waves. The last section will give an estimate of the product of transmission coefficients.

2.3.1 Up-coming and down-going waves

For any sub-interval of $\mathbb{R}$ in which the sound velocity is constant, the solution to the wave equation (1.1) consists of two components, i.e. $u(z,t) = U(z + c_i t) + D(z - c_i t)$ for $z \in [z_{i-1}, z_i]$. Assume we know the value of $u(z,t)$ at $\hat{t}$ for $z \in [z_{i-1}, z_i]$. The for t close to $\hat{t}$

$$U(z + c_i t) = \frac{1}{2} u(z + c_i t, \hat{t}) - \frac{1}{2c_i} \int_{z_{i-1}}^{z + c_i t} \frac{\partial u}{\partial t}(x, \hat{t}) dx$$

is called **up-coming wave**, which describes the wave traveling from the inside of the earth towards the surface.

$$D(z - c_i t) = \frac{1}{2} u(z - c_i t, \hat{t}) + \frac{1}{2c_i} \int_{z_{i-1}}^{z - c_i t} \frac{\partial u}{\partial t}(x, \hat{t}) dx$$

is called **down-going wave**, which represents the wave traveling in the direction from the surface to the deep earth.

For an interval that c is constant, the up-coming wave U and down-going wave D satisfy the strong wave equations

$$\frac{1}{c}\frac{\partial U}{\partial t} = \frac{\partial U}{\partial z}, \ \frac{1}{c}\frac{\partial D}{\partial t} = -\frac{\partial D}{\partial z}.$$

2.3.2 Reflections and transmissions of waves

When waves travel through a material with a discontinuous sound velocity, the reflection and transmission will occur at the jumps.

If c has only one jump, i.e. the material has a single interface (see Figure 2.1), the solution to the wave equation has the form

$$u(z,t) = \begin{cases} U_1(z + c_1 t) + D_1(z - c_1 t), & z \le z_1 \\ U_2(z + c_2 t) + D_2(z - c_2 t), & z > z_1 \end{cases}$$

where the U_i means the up-coming wave on the interval with velocity c_i and D_i means the down-going wave on the interval with velocity c_i.

Assume the support of initial conditions lies in $[z_1, +\infty)$ (see Figure 2.1). Then $D_1 = 0$, i.e. no down-going wave on the interval $(-\infty, z_1]$. u is a weak solution to the wave equation (1.1). By Appendix A.3, u has the following form.

If $t > 0$,

$$u(z,t) = \begin{cases} \dfrac{2c_1}{c_1 + c_2} U_2\left(\dfrac{c_2}{c_1} z + \left(1 - \dfrac{c_2}{c_1}\right) z_1 + c_2 t\right) & z \le z_1; \\ \dfrac{c_1 - c_2}{c_1 + c_2} U_2(2z_1 - z + c_2 t) + U_2(z + c_2 t) + D_2(z - c_2 t) & z \ge z_1. \end{cases} \tag{2.11}$$

Assume the layer i has sound velocity c_i. For a wave travels from the layer $i + 1$

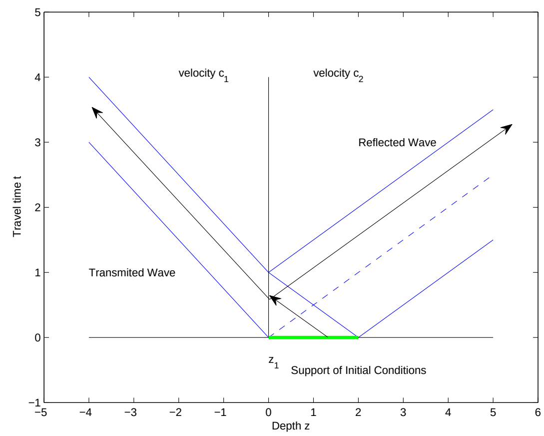

Figure 2.1 : The transmission and reflection of waves on domain with a single interface.

to the layer i, the reflection and transmission coefficients are

$$R_{i,i+1} = \frac{c_i - c_{i+1}}{c_i + c_{i+1}}, \quad T_{i,i+1} = \frac{2c_i}{c_i + c_{i+1}}.$$

2.3.3 Ray-tracing backwards in time

For waves travel between z and Z with $z < Z$, the travel time is

$$t = \int_z^Z \frac{1}{c}.$$

Assume $z \in [z_{k-1}, z_k]$ and $Z \in [z_{i-1}, z_i]$.

$$t = \frac{z_k - z}{c_k} + \sum_{j=k+1}^{i-1} \frac{z_j - z_{j-1}}{c_j} + \frac{Z - z_{i-1}}{c_i}.$$

Rearrange the above equation and get the Z as a function of (z, t), where i depends on k with $i \geq k$. Then we have

$$Z(z,t) = \frac{c_i}{c_k} z + c_i t + \sum_{j=k}^{i-1} \left(\frac{c_i}{c_{j+1}} - \frac{c_i}{c_j} \right) z_j. \tag{2.12}$$

2.3.4 The purely transmitted part of a solution

Assume the initial conditions are supported on a sub-interval $[z_{i-}, z_i]$. Part of this wave travels up and part of it travels down. Reflections and transmissions will occur when it meets a subsurface. The part of this wave which propagates all the way to the surface of the earth $(z = 0)$ without any reflection is called the **purely transmitted part of the wave**, denoted by u_T in this thesis.

For wave equations with local supported initial conditions (if the length of its support is small enough), section 3.2 of Chapter 3 of this thesis shows that the purely transmitted part of the wave arrives at the surface first and is orthogonal to the rest, which is $u - u_T$. Now consider the material with multi-interfaces and localized initial conditions (Figure 2.2).

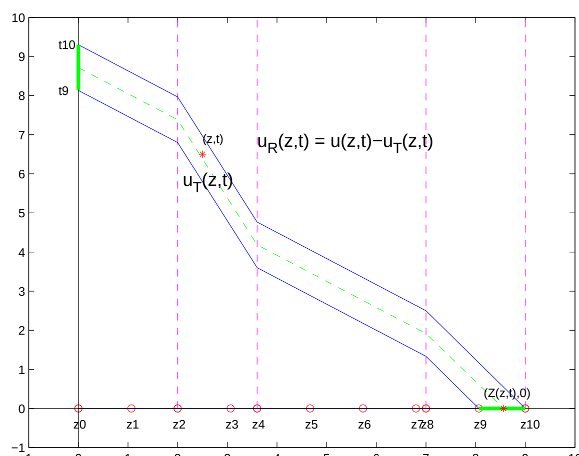

Figure 2.2 : Transmissions of waves on a domain with multi-interfaces.

The solution to the problem with multi-interfaces can be constructed by repeating the reflection and transmission. The purely transmitted solution can be given explicitly. For $[z,t]$ with $z \in [z_{k-1}, z_k]$, denote the purely transmitted solution with initial conditions supported in $[z_{i-1}, z_i]$ by u_{iT}. Let $u_{iR} = u - u_{iT}$. Assume that the corresponding velocity is c_i. By repeating the transmission, we get the purely transmitted wave.

$$u_{iT}(z,t) = \prod_{j=k}^{i-1} T_{j,j+1} U_{j+1}(Z(z,t)), \quad \text{for } t > 0, \tag{2.13}$$

$$u_{iT}(z,t) = \prod_{j=k}^{i-1} T_{j,j+1} D_{j+1}(Z(z,-t)), \quad \text{for } t < 0, \tag{2.14}$$

The constant in front of U_{j+1} and D_{j+1} are the product of transmission coefficients.

2.3.5 The product of transmission coefficients

I will give an positive lower bound of the product of transmission coefficients. This lower bound will relate the lower bound of the acoustic transparency for piecewise constant medium with the total variation of the wave velocity c.

The transmission coefficient $T_{i,i+1}$ describes the change of amplitude of the waves traveling from the layer with velocity c_{i+1} to the layer with velocity c_i. The following lemma relates the product of transmission coefficients with the variation of $c(z)$.

Lemma 2.5

$$\prod_{i=1}^{N-1} T_{i,i+1} \geq \exp\left(-\frac{\hat{r}}{2}\mathrm{Var}(\log c)\right). \tag{2.15}$$

for some constant $\hat{r}$, depending on the maximum and minimum of c.

The proof of this lemma given below is new as for as I know.

Proof. Assume $c_{\min}$ and $c_{\max}$ be the lower and upper bounds of c. Let $T_{i,i+1} = \dfrac{2c_i}{c_i + c_{i+1}}$ be the transmission coefficient and $R_{i,i+1} = 1 - T_{i,i+1}$ be the reflection coefficient. It follows that

$$0 < m = \frac{2c_{\min}}{c_{\min} + c_{\max}} \leq T_{i,i+1} \leq M = \frac{2c_{\max}}{c_{\min} + c_{\max}}$$

with $m \leq 1 \leq M$.

Approximate log from below by linear functions on the intervals $[m, 1]$ and $[1, M]$: if $T_{i,i+1} \in [m, 1]$, then

$$\log T_{i,i+1} \geq (T_{i,i+1} - 1)[(-\log m)/(1 - m)],$$

whereas if $T_{i,i+1} \in [1, M]$ then

$$\log T_{i,i+1} \geq (T_{i,i+1} - 1)[(\log M)/(M - 1)].$$

the quantities in brackets being positive in both cases. Since $R_{i,i+1} = 1 - T_{i,i+1}$, this amounts to

$$\log T_{i,i+1} \geq \begin{cases} -R_{i,i+1}[(-\log m)/(1 - m)] & \text{if} \quad T_{i,i+1} \leq 1, \\ -R_{i,i+1}[(\log M)(M - 1)] & \text{if} \quad T_{i,i+1} \geq 1 \end{cases}$$

Since $R_{i,i+1} \geq 0$ in the first case and $R_{i,i+1} \leq 0$ in the second, in both cases

$$\log T_{i,i+1} \geq -\hat{r}|R_{i,i+1}|, \text{ where } \hat{r} = \max\left(\frac{-\log m}{1 - m}, \frac{\log M}{M - 1}\right). \tag{2.16}$$

Consequently

$$\log \prod_{i=1}^{N-1} T_{i,i+1} \geq -\hat{r} \sum_{i=1}^{N-1} |R_{i,i+1}| = -\hat{r} \sum_{i=1}^{N-1} \left| \frac{1 - r_i}{1 + r_i} \right|$$

in which $r_i = c_i/c_{i+1} > 0$. Using inequality $\dfrac{|1-x|}{|1+x|} \leq \dfrac{1}{2}|\log x|, x > 0$,

$$-\hat{r}\sum_{i=1}^{N-1}\left|\frac{1-r_i}{1+r_i}\right| \geq -\frac{\hat{r}}{2}\sum_{i=1}^{N-1}|\log r_i| = -\frac{\hat{r}}{2}\mathrm{Var}(\log c),$$

whence

$$\prod_{i=1}^{N-1} T_{i,i+1} \geq \exp\left(-\frac{\hat{r}}{2}\mathrm{Var}(\log c)\right). \tag{2.17}$$

Chapter 3

Proof of the Acoustic Transparency Theorem

This chapter will give a proof of the acoustic transparency theorem.

Theorem 3.1 (Acoustic transparency theorem)

Suppose the initial data $u_0 \in H^1_{loc}(\mathbb{R})$ *and* $u_1 \in L^2_{loc}(\mathbb{R})$ *are supported on* $[0, Z]$ *and* $T = \int_0^Z \frac{1}{c}$. *Assume* $c \in BV(0, Z)$, $c(z) = c(0)$ *if* $z < \epsilon$ *and* $c(z) = c(Z)$ *if* $z > Z - \epsilon$ *with a small* $\epsilon > 0$, *i.e assume absorbing boundary conditions for domain* $[0, Z]$. *Also assume* $0 < c_{\min} \leq c \leq c_{\max}$. *Then*

$$k\|[u_0, u_1]\|^2 \leq \|W[u_0, u_1]\|^2 \leq K\|[u_0, u_1]\|^2 \tag{3.1}$$

with $K = c_{\max}$ *and*

$$k = \frac{c^2_{\min}}{2c_{\max}} \left(1 + Var(\text{logc})\exp(2\,Var(\text{logc}))\right)^{-1}. \tag{3.2}$$

The upper bound K is from lemma 2.3. The lower bound for wave equations with bounded variation coefficients is obtained using sideways energy analysis, (see Symes [13], Lewis and Symes [9]), which gives the complete proof of the acoustic transparency theorem. Please see section 3.1. A lower bound for wave equations with piecewise constant coefficients is also given using the discussion of the purely transmitted waves in chapter 2.

First we prove the upper bound. By lemma 2.3, let $t = T$ and rearrange equation (2.8) and get

$$\begin{aligned}
& \frac{1}{c(0)} \int_0^T \left(\frac{\partial u}{\partial t}(0,t) \right)^2 dt + \frac{1}{c(Z)} \int_0^T \left(\frac{\partial u}{\partial t}(Z,t) \right)^2 dt \\
= \;& E(0) - E(T) \leq E(0) \\
\leq \;& \frac{1}{2} \int_0^Z \left(\left(\frac{\partial u}{\partial z} \right)^2 + \frac{1}{c^2} \left(\frac{\partial u}{\partial t} \right)^2 \right) dz \\
= \;& \frac{1}{2} \int_0^Z \left(\left(\frac{du_0}{dz} \right)^2 + \frac{u_1^2}{c^2} \right) dz
\end{aligned}$$

This gives us the upper bound of our theorem with $K = c_{\max}$.

The next section will give a proof of the lower bound of the acoustic transparency theorem.

3.1 A proof of the acoustic transparency theorem by sideways energy estimates

This section gives a complete proof of the lower bound of the acoustic transparency theorem.

I need the Gronwall's inequality in the proof. Thus I state it here. See page 63 lemma 2 of [1].

Lemma 3.1 (**Gronwall's inequality**)

Assume that $P, f : [a, b) \to \mathbb{R}$ *are two continuous functions. If* f *is non-negative and* P *satisfies*

$$P(z) \leq \alpha(z) + \int_a^z f(s)P(s)ds$$

for $z > a$, then if $\alpha(z)$ is non-decreasing, P satisfies

$$P(z) \leq \alpha(z)\exp\left(\int_a^z f(s)ds\right)$$

3.1.1 Wave equations with smooth coefficients

Assume the density of the material is smooth. Denote it by ρ. Also suppose that the wave velocity c is smooth. Then the solution to the wave equation is smooth if the initial data is also smooth. Define

$$\tau(z) = \int_0^z \frac{1}{c} dz$$

as the travel time between 0 and z.

This subsection deals with wave equations with smooth coefficients. For the wave equation (1.1), let

$$p = -\rho\frac{\partial u}{\partial t}, v = \frac{\partial u}{\partial z}.$$

Then (p, v) satisfies wave equations (with $\kappa = \rho c^2$)

$$\frac{1}{\kappa}\frac{\partial p}{\partial t} = -\frac{\partial v}{\partial z}, \rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z}.$$

First, define the **sideways energy** following Symes [14] as

$$P(z) = \frac{1}{2}\int_{-T+\tau(z)}^{T-\tau(z)} \left(\frac{p^2}{\kappa} + \rho v^2\right)(z,t)dt \tag{3.3}$$

By differentiating $P(z)$,

$$\begin{aligned}\frac{dP}{dz}(z) &= -\frac{1}{2c(z)}\left[\left(\frac{p^2}{\kappa} + \rho v^2\right)(z, T - \tau(z)) + \left(\frac{p^2}{\kappa} + \rho v^2\right)(z, -T + \tau(z))\right] \\ &\quad + \int_{-T+\tau(z)}^{T-\tau(z)} \left[\frac{1}{\kappa}p\frac{\partial p}{\partial z} + \rho v\frac{\partial v}{\partial z} - \frac{1}{2}\left(\frac{d\kappa}{dz}\frac{1}{\kappa}\right)\frac{p^2}{\kappa} + \frac{1}{2}\left(\frac{d\rho}{dz}\frac{1}{\rho}\right)\rho v^2\right] dt\end{aligned}$$

Substitute the wave equations $\frac{1}{\kappa}\frac{\partial p}{\partial t} = -\frac{\partial v}{\partial z}$ and $\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z}$ in to the above expression of $\frac{dP}{dz}(z)$. The second term of the right hand side becomes

$$\int_{-T+\tau(z)}^{T-\tau(z)} -\frac{\rho}{\kappa}\frac{\partial(pv)}{\partial t}dt$$

Notice that

$$\begin{aligned}
& \frac{1}{2}\left(-\frac{1}{c\kappa}p^2 - \frac{\rho}{c}v^2 + 2\frac{\rho}{\kappa}pv\right) \\
= & \frac{1}{2}\left(-\frac{\sqrt{\rho}}{\kappa\sqrt{\kappa}}p^2 - \frac{\rho\sqrt{\rho}}{\sqrt{\kappa}}v^2 + 2\frac{\rho}{\kappa}pv\right) \\
= & -\frac{1}{2}\frac{\sqrt{\rho}}{\sqrt{\kappa}}\left(\left(\frac{p}{\sqrt{\kappa}}\right)^2 + (\sqrt{\rho}v)^2 - 2\frac{p}{\sqrt{\kappa}}\sqrt{\rho}v\right) \\
= & -\frac{1}{2}\frac{\sqrt{\rho}}{\sqrt{\kappa}}\left(\frac{p}{\sqrt{\kappa}} - \sqrt{\rho}v\right)^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \frac{1}{2}\left(-\frac{1}{c\kappa}p^2 - \frac{\rho}{c}v^2 - 2\frac{\rho}{\kappa}pv\right) \\
= & -\frac{1}{2}\frac{\sqrt{\rho}}{\sqrt{\kappa}}\left(\frac{p}{\sqrt{\kappa}} + \sqrt{\rho}v\right)^2
\end{aligned}$$

Using the above results, $\frac{dP}{dz}(z)$ becomes

$$\begin{aligned}
& -\frac{1}{2}\frac{\sqrt{\rho}}{\sqrt{\kappa}}\left(\frac{p}{\sqrt{\kappa}} + \sqrt{\rho}v\right)^2 (z, T-\tau(z)) \\
& -\frac{1}{2}\frac{\sqrt{\rho}}{\sqrt{\kappa}}\left(\frac{p}{\sqrt{\kappa}} - \sqrt{\rho}v\right)^2 (z, -T+\tau(z)) \\
& +\frac{1}{2}\int_{-T+\tau(z)}^{T-\tau(z)}\left[\left(-\frac{d\kappa}{dz}\frac{1}{\kappa}\right)\frac{p^2}{\kappa} + \left(\frac{d\rho}{dz}\frac{1}{\rho}\right)\rho v^2\right]dt
\end{aligned}$$

Define $D(z,t) = \frac{p}{\sqrt{\kappa}} + \sqrt{\rho}v$ and $U(z,t) = \frac{p}{\sqrt{\kappa}} - \sqrt{\rho}v$ as the down-going wave and the up-coming wave respectively. Integrating $\frac{dP}{dz}(z)$ from 0 to z gives

$$\begin{aligned}
P(z) - P(0) & = -\frac{1}{2}\int_0^z \frac{1}{c(\zeta)}(D^2(\zeta, T-\tau(\zeta)) + U^2(\zeta, -T+\tau(\zeta)))d\zeta \\
& +\frac{1}{2}\int_0^z\int_{-T+\tau(\zeta)}^{T-\tau(\zeta)}\left[\left(-\frac{d\kappa}{dz}\frac{1}{\kappa}\right)\frac{p^2}{\kappa} + \left(\frac{1}{\rho}\frac{d\rho}{dz}\right)\rho v^2\right]dtd\zeta
\end{aligned} \tag{3.4}$$

Thus

$$\begin{aligned} P(z) &\le P(0) + \frac{1}{2}\int_0^z \left[\left|\frac{1}{\kappa}\frac{d\kappa}{dz}\right| \int_{-T+\tau(\zeta)}^{T-\tau(\zeta)} \frac{p^2}{\kappa}dt + \left|\frac{1}{\rho}\frac{d\rho}{dz}\right| \int_{-T+\tau(\zeta)}^{T-\tau(\zeta)} \rho v^2 dt\right] d\zeta \\ &\le P(0) + \int_0^z \left(\left|\frac{1}{\kappa}\frac{d\kappa}{dz}\right| + \left|\frac{1}{\rho}\frac{d\rho}{dz}\right|\right) P(\zeta)d\zeta \end{aligned}$$

By Gronwall's inequality, we get for $z \in [0, Z]$

$$P(z) \le P(0)\exp\left(\mathrm{Var}(\log\kappa) + \mathrm{Var}(\log\rho)\right). \tag{3.5}$$

Now let's do more to the equality (3.4) about $P(z)$. Take $z = Z$. By definition $P(Z) = 0$ since we assume $T = \displaystyle\int_0^Z \frac{1}{c}$. From equation (3.4), we get

$$\begin{aligned} &\frac{1}{2}\int_0^Z \frac{1}{c(\zeta)}[D^2(\zeta, T-\tau(\zeta)) + U^2(\zeta, -T+\tau(\zeta))]d\zeta \\ = \;& P(0) + \frac{1}{2}\int_0^Z \int_{-T+\tau(\zeta)}^{T-\tau(\zeta)} \left(\left(-\frac{1}{\kappa}\frac{d\kappa}{dz}\right)\frac{p^2}{\kappa} + \left(\frac{1}{\rho}\frac{d\rho}{dz}\right)\rho v^2\right) dt d\zeta \\ \le \;& P(0) + \int_0^Z \left(\left|\frac{1}{\kappa}\frac{d\kappa}{dz}\right| + \left|\frac{1}{\rho}\frac{d\rho}{dz}\right|\right) P(\zeta)d\zeta \\ \le \;& P(0)\left(1 + (\mathrm{Var}(\log\kappa) + \mathrm{Var}(\log\rho))\exp\left(\mathrm{Var}(\log\kappa) + \mathrm{Var}(\log\rho)\right)\right). \end{aligned} \tag{3.6}$$

Let $\zeta(T-t)$ be the position of the space, such that the travel time between 0 and $\zeta(T-t)$ is t. Define the natural energy over spatial interval $[0, \zeta(T-t)]$ as before

$$E(t) = \frac{1}{2}\int_0^{\zeta(T-t)} \left(\frac{p^2}{\kappa} + \rho v^2\right) dz.$$

Since $\dfrac{d\tau}{dz} = \dfrac{1}{c}$, it implies that $\dfrac{d\zeta}{dt}(t) = c(\zeta(t))$, which implies that

$$\frac{d}{dt}\zeta(T-t) = -c(\zeta(T-t)).$$

Differentiate $E(t)$ and get

$$\frac{dE(t)}{dt} = -\frac{1}{2}c(\zeta(T-t))\left(\frac{p^2}{\kappa} + \rho v^2\right)(\zeta(T-t), t) + \int_0^{\zeta(T-t)} \left(\frac{p}{\kappa}\frac{\partial p}{\partial t} + \rho v\frac{\partial v}{\partial t}\right) dz$$

Substituting the wave equations into the above equation, the second term of the above equation becomes $\int_0^{\zeta(T-t)} -\frac{\partial}{\partial z}(pv)dz$. Notice that

$$\begin{aligned} & \frac{1}{2}\sqrt{\frac{\kappa}{\rho}}\left(\frac{p^2}{\kappa}+\rho v^2\right)+pv \\ = \ & \frac{1}{2}\sqrt{\frac{\kappa}{\rho}}\left(\frac{p}{\sqrt{\kappa}}+\sqrt{\rho}v\right)^2. \end{aligned}$$

Thus, with $D = \frac{p}{\sqrt{\kappa}} + \sqrt{\rho}v$ be the down going wave, we have

$$\frac{dE(t)}{dt} = pv(0,t) - \frac{1}{2}\sqrt{\frac{\kappa}{\rho}}D^2(\zeta(T-t),t).$$

Integration from 0 to t gives

$$E(t) = E(0) + \int_0^t pv(0,\tau)d\tau - \int_0^t \frac{c(\zeta(T-t))}{2}D^2(\zeta(T-t),t)dt.$$

Let $t = T$. $E(T) = 0$. By Cauchy-Schwarz inequality,

$$\int_0^T pv(0,t)dt \leq \frac{c(0)}{2}\int_0^T \left(\frac{p^2}{\kappa}+\rho v^2\right)(0,t)dt \leq c(0)P(0).$$

Then do the change of variable for the third term of the expression $E(t)$. Change the variable t with $\zeta(T-t)$. $dt = -\frac{d\zeta}{c(\zeta)}$. The third term becomes

$$\frac{1}{2}\int_0^Z D^2(\zeta, T-\tau(\zeta))d\zeta.$$

Thus get the inequality

$$E(0) - \frac{1}{2}\int_0^Z D^2(\zeta, T-\tau(\zeta))d\zeta \leq c(0)P(0). \tag{3.7}$$

Define $k_1 = c(0) + c_{\max}\left(1 + (\mathrm{Var}(\log\kappa) + \mathrm{Var}(\log\rho))\exp\left(\mathrm{Var}(\log\kappa) + \mathrm{Var}(\log\rho)\right)\right)$. Equations (3.7) and (3.6) give the result

$$E(0) \leq k_1 P(0), \tag{3.8}$$

with k_1 defined above.

Now substitute the relation between the solution to first order wave equations and the solution to second order wave equations:

$$p = -\rho \frac{\partial u}{\partial t}, v = \frac{\partial u}{\partial z}.$$

Also assume $\rho = 1$. Then inequality $E(0) \leq k_1 P(0)$ becomes

$$\frac{1}{2}\int_0^Z \left(\left(\frac{du_0}{dz} \right)^2 + \frac{u_1^2}{c^2} \right) dz \leq k_1 \frac{1}{2} \int_{-T}^{T} \left(\frac{1}{c^2} \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right) (0, \cdot) dz$$

Since c is constant for $z < \epsilon$, thus $\frac{\partial u}{\partial z}(0, \cdot) = \pm \frac{1}{c} \frac{\partial u}{\partial t}(0, \cdot)$. Thus the lower bound for second order wave equation is

$$k = k_1^{-1} c_{\min}^2.$$

3.1.2 Wave equations with bounded variation coefficients

I will focus on the second order wave equations with bounded variation coefficients in this section. In order for simplicity, I assume the density $\rho = 1$ in this section.

By Lemma 2.4, there exists a sequence of smooth functions c_k such that

$$\int_0^Z |c_k - c| dz \to 0, \ \mathrm{Var}(c_k) \to \mathrm{Var}(c).$$

Thus

$$\int_0^Z \left| \frac{1}{c_k^2} - \frac{1}{c^2} \right| dz \leq C \int_0^Z |c_k - c| dz \to 0.$$

Here $C = \frac{2c_{\max}}{c_{\min}^4}$. Thus there is a subsequence of c_k (using the same notation), such that $\frac{1}{c_k^2} \to \frac{1}{c^2}$ almost everywhere.

Then for $u \in L^2[0, Z]$,

$$\int_0^Z \left| \frac{u}{c_k^2} - \frac{u}{c^2} \right|^2 dz \to 0.$$

According to the bi-linear forms (2.5), define the bi-linear forms for wave equations with coefficients c_k as

$$a_k(u, v) = a(u, v),\ b_k(u, v) = \int_{\mathbb{R}} \frac{1}{c_k^2} uvdz$$

Since c is constant outside of $[\epsilon, Z - \epsilon]$, let $c_k(z) = c(z)$ for z outside of $[\epsilon, Z - \epsilon]$. Define operator $B_k, B : L^2[-M, M] \to L^2[-M, M]$ as

$$B_k u = \frac{u}{c_k^2}, Bu = \frac{u}{c^2},$$

which satisfy

$$b_k(u, v) = \langle B_k u, v \rangle, b(u, v) = \langle Bu, v \rangle.$$

Then we have

$$\|B_k u - Bu\|^2_{L^2[-M,M]} = \int_0^Z \left| \frac{u}{c_k^2} - \frac{u}{c^2} \right|^2 dz \to 0$$

as k goes to $+\infty$.

Assume that u_k solves the wave equation (1.1) with a coefficient c_k and the same initial conditions u_0 and u_1. Then apply Lemma 2.1 (Theorem 2.8.2 of Stolk [12]) and get that

$$u_k \to u$$

in $C([-T, T], H_0^1[0, Z]) \cap C^1([-T, T], L^2[0, Z])$, which gives that for a fixed t

$$\int_0^Z \left(\frac{\partial u_k}{\partial z} - \frac{\partial u}{\partial z} \right)^2 (z, t)dz \to 0,\ \int_0^Z \left(\frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t} \right)^2 (z, t)dz \to 0.$$

Since u and u_k satisfy the wave equation (1.1) with coefficients c and c_k respectively, we have $u - u_k$ satisfies

$$\frac{1}{c^2}\frac{\partial^2(u-u_k)}{\partial t^2} - \frac{\partial^2(u-u_k)}{\partial z^2} = \left(\frac{1}{c_k^2} - \frac{1}{c^2}\right)\frac{\partial^2 u_k}{\partial t^2}$$

Since u_k is smooth, the right hand side goes to 0 in $L^1[0, Z]$ as $k \to +\infty$. the initial condition is $u - u_k = 0$. Thus by the same process of the proof in section 2.1.2, consider the zero initial value problem on $\mathbb{R}^2$, with $w = \eta u$ and $\eta \in C^\infty(\mathbb{R})$ with $\eta(z) = 1$ if $z \leq 0$ and $\eta(z) = 0$ if $z \geq \epsilon > 0$.

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = 2\frac{d\eta}{dz}\frac{\partial(u-u_k)}{\partial z} + \frac{d^2\eta}{dz^2}(u-u_k) + \eta\left(\frac{1}{c_k^2} - \frac{1}{c^2}\right)\frac{\partial^2 u_k}{\partial t^2}$$

By the same analysis of the proof in section 2.1.2, get

$$\int_{-T}^{T}\left(\frac{\partial u_k}{\partial t} - \frac{\partial u}{\partial t}\right)^2(0,t)dz \to 0.$$

Define $E_k(0)$ as the energy with sound velocity c_k and u replaced by u_k. And also define $P_k(0)$ as the net power transmitted through $z = 0$ over the time $[-T, T]$ with u_k. From the results that $u_k(t), u(t) \in L^2(\mathbb{R})$ and $\frac{\partial u_k}{\partial t}, \frac{\partial u}{\partial t} \in L^2(\mathbb{R})$, and $\frac{\partial u}{\partial t}(0, \cdot) \in L^2[-T, T]$,

$$E_k(0) \to E(0),\ P_k(0) \to P(0).$$

By lemma 2.4, we also have that

$$\mathrm{Var}(c_k) \to \mathrm{Var}(c).$$

This implies that for wave equations with bounded variation coefficients c,

$$P(0) \geq kE(0),$$

with $k = \dfrac{c_{\min}^2}{2c_{\max}}\left(1 + \mathrm{Var}(\mathrm{logc})\exp(2\mathrm{Var}(\mathrm{logc}))\right)^{-1}$.

Thus get the lower bound for wave equations with bounded variation coefficients.

3.2 Acoustic transparency for piecewise constant material

From now on, assume that c is piecewise constant. Piecewise constant function space is a subset of bounded variation function space. Thus the acoustic transparency theorem is also true for piecewise constant material.

A different approach for piecewise constant coefficients is worth to mention for two reasons. First, the proof given in section 3.1 relies on the travel time function and its inverse. It is hard to generalize the argument into multidimensional problem. This different approach gives us hope to generalize into multidimensional problem. Second, the different analysis hints that the bounded variation assumption may not be necessary for acoustic transparency theorem.

Theorem 3.2 (Acoustic transparency theorem for piecewise constant coefficients)

Suppose the initial data $u_0 \in H^1_{loc}(\mathbb{R})$ and $u_1 \in L^2_{loc}(\mathbb{R})$ are supported on $[0, Z]$ and $T > \int_0^Z \frac{1}{c}$. Assume c is piecewise constant, $c(z) = c(0)$ if $z < \epsilon$ and $c(z) = c(Z)$ if $z > Z - \epsilon$ with a small $\epsilon > 0$, i.e assume absorbing boundary conditions for domain $[0, Z]$. Also assume $0 < c_{\min} \le c \le c_{\max}$. Then

$$\|W[u_0, u_1]\|^2 \ge e^{-N\alpha}\frac{k}{2}\|[u_0, u_1]\|^2 \tag{3.9}$$

with $\alpha = \ln\left(\frac{c_{\max}}{k}\right)$,

$$k = \frac{c_{\min}}{c_{\max}} \exp\left(-\hat{r}\, Var(\text{logc})\right). \tag{3.10}$$

$\hat{r}$ is given by equation (2.16) and $N+1$ is the number of points in a fixed finer partition given by equation 3.11 (N depends on c).

The discontinuous points of c forms a partition of the interval $[0, Z]$. Let h be the minimum length between two successive jumps of c. Then refine the partition formed by the discontinuous points of c and get a finer partition $\{z_0, z_1, \cdots, z_N\}$ with $z_0 = 0$, $z_N = Z$ and

$$|z_i - z_{i-1}| \leq \frac{2c_{\min}}{c_{\max}} h. \tag{3.11}$$

See figure 3.1 for the finer partition.

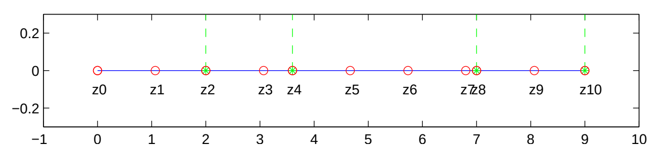

Figure 3.1 : A finer partition of $[0, Z]$ according to the jumps of c.

Why do we need this finer partition? Please see figure 3.2 and figure 3.3.

The finer partition can make sure that for initial conditions that are supported on a sub-interval, the solution to the wave equation on the surface over a corresponding time interval contains only the purely transmitted part of the solution. If we do not refine the partition, the solution over a corresponding time interval may contain also the reflected waves, as is shown in figure 3.2.

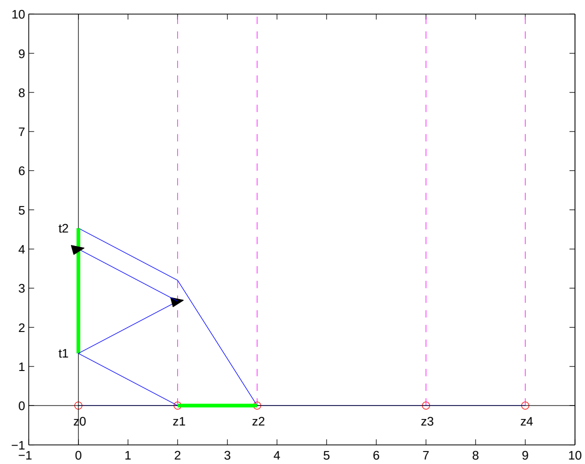

Figure 3.2 : The purely transmitted part of the solution restricted on the surface is NOT orthogonal to the part of the wave with at least one reflection.

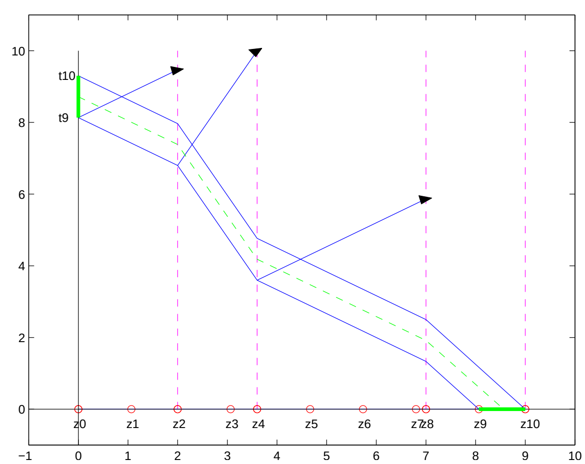

Figure 3.3 : The purely transmitted part of the solution restricted on the surface IS orthogonal to the part of the wave with at least one reflection.

Since $\{z_0, z_1, \cdots, z_N\}$ is a finer partition, c is also a step function on this partition. Let $t_i = \int_0^{z_i} \frac{1}{c}$ be the travel time from $z = 0$ to $z = z_i$. Denote by c_i the wave velocity in the sub-interval $[z_{i-1}, z_i]$. Two successive c_i may be the same due to the refinement of the partition.

The length of the finer partition gives that

$$|t_i - t_{i-1}| = \frac{|z_i - z_{i-1}|}{c_i} \leq 2\frac{h}{c_{\max}}.$$

This means that the length of each time interval is less than the time required for the wave to travel up and be reflected back through any of these sub-intervals. See figure 3.3. Thus we conclude that the purely transmitted wave u_{iT} arrives at each jump as well as the surface first and is orthogonal to the rest, which is $u_{iR} = u - u_{iT}$.

Denote by $H^{(1)}[z_{i-1}, z_i] = \{u(z_{i-1}) = u_0(z_{i-1}) : u \in H^1[z_{i-1}, z_i]\}$. Decompose the spaces of initial conditions as

$$(u_0, u_1) \in H^1[0, Z] \times L^2[0, Z] = \prod_{i=1}^{N} \left(H^{(1)}[z_{i-1}, z_i] \times L^2[z_{i-1}, z_i]\right)$$

Correspondingly, decompose the trace space $L^2[-T, T]$ as

$$\frac{\partial u}{\partial t}(0, t) \in L^2[-T, T] = \prod_{i=1}^{N+1} \left(L^2[t_{i-1}, t_i] \times L^2[-t_i, -t_{i-1}]\right),$$

with $t_{N+1} = T$ since $T > \int_0^Z \frac{1}{c}$.

Assume the support of u_0 and u_1 is a subset of $[z_{i-1}, z_i]$. The corresponding velocity on this interval is $c = c_i$. For t small, the up-coming wave on $[z_{i-1}, z_i]$ can be written as

$$U_i(z + c_i t) = \frac{1}{2}u_0(z + c_i t) - \frac{1}{2c_i}\int_0^{z+c_i t} u_1(x)dx. \tag{3.12}$$

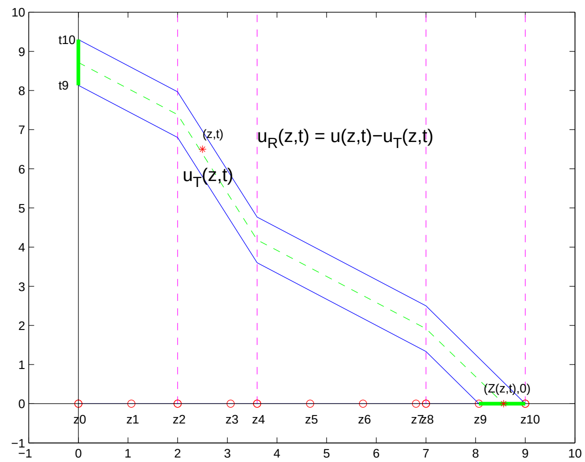

Figure 3.4 : Transmissions of wave on domain with multi-interfaces.

The down-going wave on $[z_{i-1}, z_i]$ can be written as

$$D_i(z - c_i t) = \frac{1}{2}u_0(z - c_i t) + \frac{1}{2c_i}\int_0^{z-c_i t} u_1(x)dx. \tag{3.13}$$

The subscript T denote the part of the solution with only transmissions and R with at least one reflection. For problem with initial conditions supported in $[0, Z]$, let

$$u_i(z,t) = u_{iT}(z,t) + u_{iR}(z,t)$$

be the solution with initial conditions $u_0\chi_{[z_{i-1},z_i]}$ and $u_1\chi_{[z_{i-1},z_i]}$. From equations (2.13) and (2.14), here u_{iT} has the form

$$u_{iT}(z,t) = \prod_{j=k}^{i-1} T_{j,j+1}U_{j+1}(Z(z,t)), \quad \text{for } t > 0, \tag{3.14}$$

$$u_{iT}(z,t) = \prod_{j=k}^{i-1} T_{j,j+1}D_{j+1}(Z(z,-t)), \quad \text{for } t < 0, \tag{3.15}$$

Then the solution to the wave equation (1.1) with arbitrary initial conditions is the sum of these u_i

$$u(z,t) = \sum_{i=1}^{N} u_i(z,t).$$

The purely transmitted part of the solution can be written as

$$u_T(z,t) = \sum_{i=1}^{N} u_{iT}(z,t).$$

The support of u_{iT} is $[-t_i, -t_{i-1}] \cup [t_{i-1}, t_i]$. Thus $u_{iT}, i = 1, \cdots, N$ are orthogonal to each other. Let $T_{k,k+1} = \dfrac{2c_k}{c_k + c_{k+1}}$ and consider $T > \displaystyle\int_0^Z \frac{1}{c(z)}$. Together with equations (3.12) and (3.13) We have

$$\begin{aligned}
&\int_{-T}^{T} \left(\frac{\partial u_T}{\partial t}(0,t) \right)^2 dt \\
&\geq \sum_{i=1}^{N} \left(\int_{-t_i}^{-t_{i-1}} + \int_{t_{i-1}}^{t_i} \right) \left(\frac{\partial u_{iT}}{\partial t}(0,t) \right)^2 dt \\
&= \sum_{i=1}^{N} \left(\prod_{k=1}^{i-1} \frac{2c_k}{c_k + c_{k+1}} \right)^2 \left(\int_{t_{i-1}}^{t_i} \left(\frac{\partial u_{iU}}{\partial t}(Z(0,t)) \right)^2 dt + \int_{-t_i}^{t_{i-1}} \left(\frac{\partial u_{iD}}{\partial t}(Z(0,-t)) \right)^2 dt \right) \\
&= \sum_{i=1}^{N} \left(\prod_{k=1}^{i-1} T_{k,k+1} \right)^2 \frac{c_i^2}{4c_1} \int_{z_{i-1}}^{z_i} \left(\left(\frac{du_0}{dz} \right)^2 + \frac{u_1(z)^2}{c_i^2} \right) dz
\end{aligned} \tag{3.16}$$

Use this result and lemma 2.5. There exists a constant

$$k = \frac{c_{\min}}{c_{\max}} \exp\left(-\hat{r}\mathrm{Var}(\log c)\right), \tag{3.17}$$

such that

$$\int_{-T}^{T} \left(\frac{\partial u_T}{\partial t}(0,t) \right)^2 dt \geq k \int_0^Z \left(\frac{du_0}{dz}^2 + \frac{u_1^2}{c^2} \right) dz. \tag{3.18}$$

According to equations (3.12), (3.13), (2.13) and (2.14), define a linear operator $\mathcal{T}_i$ as

$$\frac{\partial u_{iT}}{\partial t}(0,t) = \mathcal{T}_i[u_0 \chi_{[z_{i-1},z_i]}, u_1 \chi_{[z_{i-1},z_i]}].$$

And then define $\mathcal{T}_D = \text{diag}[\mathcal{T}_1, \cdots, \mathcal{T}_N]$ which is a linear operator from $H^1[0, Z] \times L^2[0, Z]$ to $L^2[-T, T]$

$$\frac{\partial u_T}{\partial t}(0, t) = \mathcal{T}_D[u_0, u_1].$$

Since

$$H^1[0, Z] \times L^2[0, Z] = \prod_{i=1}^{N} \left(H^{(1)}[z_{i-1}, z_i] \times L^2[z_{i-1}, z_i]\right)$$

and

$$L^2[-T, T] = \prod_{i=1}^{N} \left(L^2[t_{i-1}, t_i] \times L^2[-t_i, -t_{i-1}]\right),$$

define linear operator $W_z : H^1[0, Z] \times L^2[0, Z] \to H^1[0, Z] \times L^2[0, Z]$ as

$$W_z = \text{diag}[e^{-\alpha} I, \cdots, e^{-N\alpha} I],$$

where I is identity operator from $\left(L^2[t_{i-1}, t_i] \times L^2[-t_i, -t_{i-1}]\right)$ to itself. And define linear operator $W_t : L^2[-T, T] \to L^2[-T, T]$ as

$$W_t = \text{diag}[e^{-\alpha} I, \cdots, e^{-N\alpha} I],$$

where I is identity operator from $H^{(1)}[z_{i-1}, z_i] \times L^2[z_{i-1}, z_i]$ to itself.

Define the following norms on $H^1[0, Z] \times L^2[0, Z]$ and $L^2[-T, T]$ as

$$\|[u_0, u_1]\|^2 = \int_0^Z \left(\left(\frac{du_0}{dz}\right)^2 + \left(\frac{u_1}{c^2}\right)^2\right) dz,$$

$$\|[u_0, u_1]\|_w^2 = \sum_{i=1}^{N} e^{-i\alpha} \int_{z_{i-1}}^{z_i} \left(\left(\frac{du_0}{dz}\right)^2 + \left(\frac{u_1}{c^2}\right)^2\right) dz,$$

$$\left\|\frac{\partial u}{\partial t}(0, t)\right\|^2 = \int_{-T}^{T} \left(\frac{\partial u}{\partial t}(0, t)\right)^2 dt,$$

$$\left\| \frac{\partial u}{\partial t}(0,t) \right\|_w^2 = \sum_{i=1}^{N} e^{-i\alpha} \left(\int_{-t_i}^{-t_{i-1}} + \int_{t_{i-1}}^{t_i} \right) \left(\frac{\partial u}{\partial t}(0,t) \right)^2 dt,$$

where the subscript w means weighted norm. It is easy to see that $\|[u_0, u_1]\|$ is equivalent to $\|[u_0, u_1]\|_w$ and $\left\| \frac{\partial u}{\partial t}(0,t) \right\|$ is equivalent to $\left\| \frac{\partial u}{\partial t}(0,t) \right\|_w$ for a fixed $\alpha \neq 0$.

Define linear operator $\mathcal{T} : H^1[0,Z] \times L^2[0,Z] \to L^2[-T,T]$ as $\frac{\partial u}{\partial t}(0,t) = \mathcal{T}[u_0, u_1]$ and let $\mathcal{T}_L = \mathcal{T} - \mathcal{T}_D$. Let

$$\hat{u}_0 = \sum_{i=1}^{N} e^{-i\alpha} u_0 \chi_{[z_{i-1}, z_i]},$$

$$\hat{u}_1 = \sum_{i=1}^{N} e^{-i\alpha} u_1 \chi_{[z_{i-1}, z_i]},$$

$$\frac{\partial \hat{u}}{\partial t}(0,t) = \sum_{i=1}^{N} e^{-i\alpha} \frac{\partial u}{\partial t}(0,t)(\chi_{[-t_i, -t_{i-1}]} + \chi_{[t_{i-1}, t_i]}).$$

Then we have

$$W_t \mathcal{T}_D W_z^{-1}[\hat{u}_0, \hat{u}_1] + W_t \mathcal{T}_L W_z^{-1}[\hat{u}_0, \hat{u}_1] = \frac{\partial \hat{u}}{\partial t}(0,t). \tag{3.19}$$

As we said before, the trace of the solutions at the surface involving at least one reflection have support above the support of the purely transmitted part of solutions. Thus the linear operator $\mathcal{T}_L$ is a strictly lower block triangular matrix. Then there is a linear operator H, such that

$$W_t \mathcal{T}_L W_z^{-1} = e^{-\alpha} H.$$

And linearly, $W_t \mathcal{T}_D W_z^{-1} = S_D$. Then equation (3.19) gives

$$\|\mathcal{T}_D[\hat{u}_0, \hat{u}_1]\|^2 - e^{-\alpha} \|H[\hat{u}_0, \hat{u}_1]\|^2 \leq \left\| \frac{\partial \hat{u}}{\partial t}(0,t) \right\|^2,$$

which gives

$$\|\mathcal{T}_D[u_0, u_1]\|_w^2 - e^{-\alpha}\|H[u_0, u_1]\|_w^2 \leq \left\|\frac{\partial u}{\partial t}(0,t)\right\|_w^2.$$

Since $\|\mathcal{T}_D[u_0, u_1]\|^2 \geq k\|[u_0, u_1]\|^2$, we have $\|\mathcal{T}_D[u_0, u_1]\|_w^2 \geq k\|[u_0, u_1]\|_w^2$. Thus get

$$k\|[u_0, u_1]\|_w^2 - e^{-\alpha}\|H[u_0, u_1]\|_w^2 \leq \left\|\frac{\partial u}{\partial t}(0,t)\right\|_w^2.$$

By the upper bound of Theorem 3.1, $\|H[u_0, u_1]\|$ is bounded above by the upper bound of $\left\|\frac{\partial u}{\partial t}(0,t)\right\|$, i.e. $\|H(u_0, u_1)\|_w^2 \leq c_{\max}\|[u_0, u_1]\|_w^2$. Choose $\alpha = \ln\frac{2c_{\max}}{k}$. Then get

$$\left\|\frac{\partial u}{\partial t}(0,t)\right\|_w^2 \geq \frac{k}{2}\|[u_0, u_1]\|_w^2.$$

By the equivalence of the norms and consider the solutions with local supported initial conditions,

$$\left\|\frac{\partial u}{\partial t}(0,t)\right\|^2 \geq e^{-N\alpha}\frac{k}{2}\|[u_0, u_1]\|^2. \tag{3.20}$$

This equation together with equation (3.17) gives the lower bound for piecewise constant coefficient c.

Chapter 4

Discussion

4.1 Bounded variation may not be necessary

Consider a special example of the piecewise constant material. Please see figure 2.2.

Assume the material has N layers. The initial conditions are supported at the last layer $[z_{N-1}, z_N]$. By examining the expression of u_{iT}, for $t \in [t_{N-1}, t_N]$

$$u_{NT}(0,t) = \prod_{i=1}^{N-1} T_{i,i+1} U_N(Z(0,t))$$

is actually the trace on $z = 0$ of the entire solution of a wave equation. Thus $\prod_{i=1}^{N-1} T_{i,i+1}$ is an upper bound for the acoustic transparency of the material.

Lemma 2.5 shows that if $c \in BV[0, Z]$, the product of transmission coefficients

$$\prod_{i=1}^{N-1} T_{i,i+1} \geq \exp\left(-\frac{\hat{r}}{2}\mathrm{Var}(\log c)\right).$$

with $\hat{r}$ depends on the supremum and infimum of c.

Assume that N is even. Let velocity c be a periodic step function over these layers, with $c = 1 - \frac{1}{\sqrt{N}}$ if it is odd layer and $c = 1 + \frac{1}{\sqrt{N}}$ if it is even. Then we have

$$\lim_{N\to+\infty} \prod_{i=1}^{N-1} T_{i,i+1} = \lim_{N\to+\infty} \left(1 - \frac{1}{N}\right)^{\frac{N}{2}} = \exp(-\frac{1}{2}) > 0.$$

However, $\mathrm{Var}(c) = 2\sqrt{N} \to +\infty$ as $N \to +\infty$. The set of c with given positive lower bound of $\prod_{i=1}^{N-1} T_{i,i+1}$ is not bounded in $BV(0, Z)$.

The analysis of purely transmitted waves only gives a lower bound that depends on the number of layers of the material for wave velocities of piecewise constant. The sideways energy estimate could give the lower bound for wave velocities with bounded variation. However, it uses the inverse function of $\tau(z) = \int_0^z \frac{1}{c}$. For multidimensional case, the travel time between two positions may have different values. Thus it is possible that a travel time is not a function of space. Even if for some special case that the travel time is a function of space, the inverse function of this travel time function does not exist. Thus it is not obvious to find a way to generalize this analysis into multidimensional problem now.

4.2 Inverse problem

In order to get the coefficient of the wave equation (1.1) from the data measured at the surface of a material, an inverse problem is to be solved. Assume the measured data is acoustic pressure. Denote the measured data as *data*.

Define $\mathcal{F}[c] = -\rho \frac{\partial u}{\partial t}(0, \cdot)$, which is called forward map: solving for the solution of wave equation (1.1) with given c and initial data u_0, u_1.

The velocity c is recovered by solving the inverse problem

$$\min_c \|\mathcal{F}[c] - data\|.$$

Bamberger et al. [2] showed that for piecewise constant c, if the layers have equal travel time (see Goupillaud [7]) and the boundary condition is Neumann, the solution to the inverse problem is unique and depends continuously on *data*.

This thesis is a step towards a more general result: if c is a non-smooth function, the solution is also unique and depends continuously on *data*.

4.3 The relation between the acoustic transparency theorem and the data of an inverse problem

This section gives a relation between the acoustic transparency theorem and the input and output data of the inverse problem.

In this section, I assume that the coefficients and the solutions of the wave equation (1.1) are smooth.

4.3.1 Scattering operators

Assume $c = c(0) > 0$ for $z < 0$. The solution to the wave equation with a constant coefficient has the form $u(z,t) = U(z + ct) + D(z - ct)$, for all t.

Assume $\text{supp} D \subset \mathbb{R}_-$ and assume for $t < 0$, $u(z,t) = D(z - c(0)t)$, that is u travels in the direction of $z > 0$, if $t < 0$. Since $c(z) = c(0)$ for $z < \epsilon$, we have that

$$u(z,t) = U(z + c(0)t) + D(z - c(0)t),$$

for $z < 0$ and all t.

Then for $t > 0$, since $-\dfrac{1}{c(0)}\dfrac{\partial U}{\partial t} + \dfrac{\partial U}{\partial z} = 0$,

$$\left(-\frac{1}{c(0)}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z}\right)(0, \cdot) = 2D'(-ct) = g(t).$$

Define operator S by $Sg = \dfrac{1}{c(0)}\dfrac{\partial u}{\partial t}(0, \cdot)$ for $t > 0$. S is a bounded operator from $L^2(\mathbb{R}^+)$ to itself.

If $c = c(0)$ for all z, the solution has the form $u = D(z - c(0)t)$ for all z, t. Then $Sg = -D'(-c(0)t) = \dfrac{1}{2}g$, which implies that $S = \dfrac{1}{2}I$ with I denoting the identity operator.

Define operator S' by $S'g = \left(\dfrac{1}{c}\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial z}\right)(0, \cdot)$. If c is a constant for all z, $S' = 0$. In general, $S'g = 2U'(c(0)t)$.

Here operators S and S' are scattering operators.

Define operator $\Lambda g_1 = \dfrac{1}{c}\dfrac{\partial w}{\partial t}(0, \cdot)$, where w solves the wave equation (1.1) with initial conditions $w = \dfrac{\partial w}{\partial t} = 0$ for $t < 0$ and $z > 0$ and $\dfrac{\partial w}{\partial z}(z, \cdot) = g_1 \in L^2(\mathbb{R})$ and $\text{supp} g_1 \subset \mathbb{R}_+$. Here Λ is the usual Neumann to Dirichlet map for wave equations.

Then $\dfrac{\partial u}{\partial z}(0, \cdot) = \dfrac{1}{2}(g + S'g)$ and $S'g = \left(\dfrac{1}{c}\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial z}\right)(0, \cdot) = \Lambda\dfrac{1}{2}(g + S'g) + \dfrac{1}{2}(g + S'g)$. This gives that

$$(S' - I)(S' + I)^{-1} = \Lambda.$$

Equivalently,

$$S(I + S)^{-1} = \Lambda$$

Riley [11] derived a similar result in his thesis.

4.3.2 Structure of the Neumann-to-Dirichlet map for wave equations with absorbing boundary conditions

Now consider the problem with absorbing boundary conditions and non-zero initial data $[u_0, u_1]$.

Assume the initial conditions $[u_0, u_1] \in H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$. Let $T = \displaystyle\int_0^Z \frac{1}{c}$. Define operator $W : H^1(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \to L^2[-T, T]$ as before $W[u_0, u_1] = \dfrac{1}{c}\dfrac{\partial u}{\partial t}(0, \cdot)$.

Let w solves the wave equation (1.1) with initial condition $w(\cdot, 0) = \dfrac{\partial w}{\partial t}(\cdot, 0) = 0$ for $z > 0$ and a boundary condition $\dfrac{\partial w}{\partial z}(0, \cdot) = g$. Define operator X as

$$Xg = \left[w, \frac{\partial w}{\partial t}\right](\cdot, T), \text{for} z > 0.$$

The the solution w can be written as $w = q + r$ where q solves the wave equation (1.1) with $\left[q, \dfrac{\partial q}{\partial t}\right](\cdot, T) = Xg$, $\dfrac{\partial q}{\partial z}(0, \cdot) = 0$ and r solves the wave equation (1.1) with $\left[r, \dfrac{\partial r}{\partial t}\right](\cdot, T) = 0$, $\dfrac{\partial r}{\partial z}(0, \cdot) = g$.

By definition of operator Λ, $\Lambda g = \dfrac{1}{c}\dfrac{\partial w}{\partial t}(0, \cdot)$ for $t \in [0, T]$. Thus

$$\Lambda g = \frac{1}{c}\left(\frac{\partial q}{\partial t} + \frac{\partial r}{\partial t}\right)(0, \cdot).$$

Let $\tilde{r}(z, t) = r(z, T - t)$. Define $Rg(t) = g(T - t)$. Then $\left[\tilde{r}, \dfrac{\partial \tilde{r}}{\partial t}\right](\cdot, 0) = 0$ and $\dfrac{\partial \tilde{r}}{\partial z}(0, t) = g(T - t) = Rg(t)$. Thus we get

$$\frac{1}{c}\frac{\partial \tilde{r}}{\partial t}(0, \cdot) = \Lambda Rg$$

and

$$\frac{1}{c}\frac{\partial r}{\partial t}(0, \cdot) = -R\Lambda Rg. \tag{4.1}$$

By the definition of operator W and $w = q + r$, get

$$\Lambda g = WXg - R\Lambda Rg.$$

Lemma 4.1 *With the assumptions and conditions given above, the operator R and Λ satisfy $R\Lambda R = \Lambda^T$.*

Proof. By the definition of operator Λ and equation (4.1)

$$\begin{aligned}
&\langle R\Lambda Rg, \frac{\partial w}{\partial z}(0,\cdot)\rangle = -\int_0^T \frac{1}{c}\frac{\partial r}{\partial t}(0,t)\frac{\partial w}{\partial z}(0,t)dt \\
=\ & \frac{1}{c}\int_0^T\int_0^Z \frac{\partial}{\partial z}\left(\frac{\partial r}{\partial t}\frac{\partial w}{\partial z}\right)dzdt = \frac{1}{c}\int_0^T\int_0^Z \left(\frac{\partial^2 r}{\partial z\partial t}\frac{\partial w}{\partial z} + \frac{\partial r}{\partial t}\frac{\partial^2 w}{\partial z^2}\right)dzdt \\
=\ & \frac{1}{c}\int_0^T\int_0^Z \left(\frac{\partial^2 r}{\partial z\partial t}\frac{\partial w}{\partial z} + \frac{1}{c^2}\frac{\partial r}{\partial t}\frac{\partial^2 w}{\partial t^2}\right)dzdt \\
=\ & \frac{1}{c}\int_0^T\int_0^Z \frac{\partial}{\partial t}\left(\frac{\partial r}{\partial z}\frac{\partial w}{\partial z} + \frac{1}{c^2}\frac{\partial r}{\partial t}\frac{\partial w}{\partial t}\right)dzdt - \frac{1}{c}\int_0^T\int_0^Z\left(\frac{\partial r}{\partial z}\frac{\partial^2 w}{\partial z\partial t} + \frac{1}{c^2}\frac{\partial^2 r}{\partial t^2}\frac{\partial w}{\partial t}\right)dzdt \\
=\ & -\frac{1}{c}\int_0^T\int_0^Z\left(\frac{\partial r}{\partial z}\frac{\partial^2 w}{\partial z\partial t} + \frac{\partial^2 r}{\partial z^2}\frac{\partial w}{\partial t}\right)dzdt = -\frac{1}{c}\int_0^T\int_0^Z \frac{\partial}{\partial z}\left(\frac{\partial r}{\partial z}\frac{\partial w}{\partial t}\right)dzdt \\
=\ & \frac{1}{c}\int_0^T \frac{\partial r}{\partial z}(0,t)\frac{\partial w}{\partial t}(0,t)dt = \langle g, \frac{1}{c}\frac{\partial w}{\partial t}(0,t)\rangle \\
=\ & \langle g, \Lambda\frac{\partial w}{\partial z}(0,t)\rangle.
\end{aligned}$$

Thus $R\Lambda R = \Lambda^T$.

The second equality is true because $\frac{\partial w}{\partial z}(Z,t) = 0$ for $t \in [0,T]$, since $T = \int_0^Z \frac{1}{c}$. The fourth and sixth equalities are true because w and r satisfy wave equation (1.1) and

$$\frac{\partial r}{\partial z}(z,T) = \frac{\partial w}{\partial z}(z,T) - \frac{\partial q}{\partial z}(z,T) = 0$$

for $z \in [0,Z]$ by the definition of initial value problem for q. The eighth equality is true because $\frac{\partial r}{\partial z}(Z,t) = 0$ for $t \in [0,T]$ by finite propagation speed of acoustic waves.

With a similar analysis with the proof of Lemma 4.1, we get

Lemma 4.2 *With the conditions and assumptions given in this section,* $W = X^T$.

By lemma 4.1 and lemma 4.2, we have

$$\Lambda + \Lambda^T = WW^T.$$

The acoustic transparency theorem says that W^TW is a symmetric and positive definite operator. Since we have $k\|x\|^2 \leq \|Wx\|^2 \leq K\|x\|^2$ from the acoustic transparency theorem, for any Cauchy sequence $\{y_n\} \subset \mathrm{Range}(W)$, there exists $x_n \in H_0^1[0,Z] \times L^2[0,Z]$ such that $Wx_n = y_n$ and $\{x_n\}$ is also a Cauchy sequence and have a limit x. Thus $Wx = \lim_{n\to\infty} y_n$, which gives that Range(W) is closed, and we can get

$$L^2([-T,T]) = \mathrm{Range}(W) \oplus \mathrm{Kernel}(W^T).$$

For $g \in \mathrm{Range}(W)$, there is $x \in H_0^1[0,Z] \times L^2[0,Z]$, such that $g = Wx$. By acoustic transparency theorem, we have $k\|x\|^2 \leq \|Wx\|^2 = \|g\|^2 \leq K\|x\|^2$, with k and K defined in Theorem 3.1. Then

$$\left\langle g, (\Lambda + \Lambda^T)g \right\rangle = \|W^TWx\|^2 \geq k\|x\|^2 \geq \frac{k}{K}\|g\|^2,$$

i.e. $\Lambda + \Lambda^T$ is symmetric and positive definite on Range(W) and 0 on Kernel(W^T).

Appendix A

Proofs of Auxiliary Theorems

A.1 Two definitions of bounded variation functions are the same

Definition using partition of interval (see page 216 of [6]).

Definition A.1 *Let c be a Lebesgue measurable function defined on an interval (a, b). The* **essential variation** *of c on this interval is*

$$\mathrm{Var}(c) = \sup_{\mathcal{P}} \sum_{i=1}^{N} |c(z_i) - c(z_{i-1})| \, .$$

where $\mathcal{P}$ is the set of all the finite partitions of the interval (a, b). N depends on the partition. Each z_i of the partition is a point of approximate continuity of c.

Definition using integration (see page 166 of [6]).

Definition A.2 *Let $c \in L^1(a, b)$. c is a function of bounded variation in (a, b) if*

$$\sup_{\phi \in C_c^1(a,b) \ \ |\phi| \le 1} \int_a^b c\phi' dx < +\infty.$$

A space of function with a bounded variation is denoted by $BV(a, b)$. These two definitions are the same. The following proof is from page 216 of Evans and Gariepy [6].

Proof. (a). Definition A.1 $\Rightarrow$ definition A.2.

For a given $\epsilon > 0$, define $\eta_\epsilon \in C_0^\infty(\mathbb{R})$ as

$$\begin{aligned} &\eta_\epsilon(t) \ge 0 \\ &\int_{\mathbb{R}} \eta_\epsilon(t) dt = 1 \\ &\mathrm{supp}\eta_\epsilon(t) = (-\epsilon, \epsilon) \end{aligned}$$

Let $c^\epsilon = \eta_\epsilon * c$. For any $a + \epsilon < x_0 < \cdots < x_N < b - \epsilon$,

$$
\begin{aligned}
& \sum_{i=1}^{N} |c^\epsilon(z_i) - c^\epsilon(z_{i-1})| \\
= & \sum_{i=1}^{N} \left| \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s)(c(z_i - s) - c(z_{i-1} - s))ds \right| \\
\leq & \int_{-\epsilon}^{\epsilon} \eta_\epsilon(s) \sum_{i=1}^{N} |c(z_i - s) - c(z_{i-1} - s)|\, ds \\
\leq & \operatorname{Var}(c)
\end{aligned}
$$

Thus for smooth c^ϵ

$$
\int_{a+\epsilon}^{b-\epsilon} |(c^\epsilon)'|dx = \sup \left\{ \sum_{i=1}^{N} |c^\epsilon(z_i) - c^\epsilon(z_{i-1})| \right\} \leq \operatorname{Var}(c).
$$

Then for $\phi \in C_c^1(a,b)$,$|\phi| \leq 1$ and $\epsilon > 0$ small enough

$$
\int_a^b c^\epsilon \phi' dx = -\int_a^b (c^\epsilon)'\phi dx \leq \int_{a+\epsilon}^{b-\epsilon} |(c^\epsilon)'|dx \leq \operatorname{Var}(c),
$$

which gives

$$
\int_a^b c\phi' dx \leq \operatorname{Var}(c).
$$

(b). Definition A.2 $\Rightarrow$ definition A.1.

For a fixed partition $\{x_0 = a, \cdots, x_N = b\}$, define smooth functions $\eta_i^\epsilon \in C^\infty(\mathbb{R})$ as

$$
\eta_i^\epsilon(x) = \begin{cases} 1, & \text{if } x \in [x_{i-1} + \epsilon, x_i - \epsilon], \\ 0, & \text{if } x \leq x_{i-1} \text{ or } x \geq x_i. \end{cases}
$$

Choose ϕ_i such that

$$
\phi_i(x) = \operatorname{sign}(c(x_i) - c(x_{i-1}))
$$

for $x \in (x_{i-1}, x_i)$ and $\phi_i(x) = 0$ otherwise.

First, assume $c \in C^1[a,b]$.

$$
\begin{aligned}
& \sum_{i=1}^{N} |c(x_i) - c(x_{i-1})| \\
= & \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} c'(x)\phi_i(x)dx \\
= & \int_{x_{i-1}}^{x_i} c'(x) \sum_{i=1}^{N} \phi_i(x)dx \\
= & \lim_{\epsilon \to 0} \int_a^b c'(x) \sum_{i=1}^{N} (\eta_i^\epsilon \phi_i(x))\, dx \\
= & \lim_{\epsilon \to 0} \int_a^b c\phi_\epsilon' dx
\end{aligned}
$$

with $\phi_\epsilon = \sum_{i=1}^{N} \eta_i^\epsilon \phi_i(x)$.

Since c satisfies definition A.2, by section 5.2 of Evans and Gariepy [6], there exist $c_k \in C^\infty(a,b)$ such that $\int_a^b |c_k - c| dx \to 0$ and $\mathrm{Var}(c_k) \to \mathrm{Var}(c)$.

For $y, z \in (a,b)$,

$$c_k(z) = c_k(y) + \int_y^z c_k' dx.$$

Averaging with respect to $y \in (a,b)$,

$$|c_k(z)| \leq \frac{1}{b-a} \int_a^b |c_k(y)| dy + \int_a^b |c_k'| dx.$$

Thus $\|c_k\|_{L^\infty(a,b)} < \infty$ for all k, which gives

$$\|c\|_{L^\infty(a,b)} < \infty.$$

Thus if z is a point of approximate continuity of c, then

$$c_k(z) \to c(z).$$

Thus, for c satisfies definition A.2,

$$\begin{aligned} &\sum_{i=1}^{N} |c(x_i) - c(x_{i-1})| \\ \leq\; &\sum_{i=1}^{N} |c(x_i) - c_k(x_i)| + \sum_{i=1}^{N} |c_k(x_i) - c_{k-1}(x_i)| + \sum_{i=1}^{N} |c_k(x_{i-1}) - c(x_{i-1})| \end{aligned}$$

The first term and the third term go to 0 as $k \to +\infty$. The second term goes to

$$\lim_{\epsilon \to 0} \int_a^b c\phi_\epsilon' dx.$$

A.2 Smooth approximations of bounded variation functions

Proof. For convenient, denote the variation of c on interval $(0, Z)$ by $\mathrm{Var}_{(0,Z)}(c)$.

This proof follows exactly the poof of Theorem 2 on page 172 of Evans and Gariepy [6].

For a given positive integer m, define open intervals for $k = 1, \cdots$

$$U_k = \left(\frac{1}{m+k}, Z - \frac{1}{m+k}\right) \cap (-k-m, k+m).$$

For a given $\epsilon > 0$, choose m large such that $\mathrm{Var}_{(0,Z)-U_1} < \epsilon$.

Let $U_0 = 0$ and define $V_k = U_{k+1} - U_{k-1}$ for $k = 1, \cdots$. Let ζ_k with $k = 1, \cdots$ be a sequence of smooth functions such that $\zeta_k \in C_c^\infty(V_k)$, $0 \leq \zeta_k \leq 1$ and $\sum_{k=1}^{\infty} \zeta_k = 1$ on $(0, Z)$.

Define functions η_{ϵ_k} as in the proof of Lemma 2.1. For each k, choose ϵ_k small enough, such that

$$\begin{aligned} &\operatorname{supp}(\eta_{\epsilon_k} * (c\zeta_k)) \subset V_k, \\ &\int_0^Z |\eta_{\epsilon_k} * (c\zeta_k) - c\zeta_k| dz < \frac{\epsilon}{2^k}, \\ &\int_0^Z |\eta_{\epsilon_k} * (c\zeta_k') - c\zeta_k'| dz < \frac{\epsilon}{2^k}. \end{aligned} \tag{A.1}$$

Define $c_\epsilon = \sum_{k=1}^{\infty} \eta_{\epsilon_k} * (c\zeta_k)$. In a small interval of each point $z \in (0, Z)$, there are only finitely many nonzero items in this sum. Thus

$$c_\epsilon \in C^\infty(0, Z).$$

Since $c = \sum_{k=1}^{\infty} c\zeta_k$,

$$\|c - c_\epsilon\|_{L^1(0,Z)} \leq \sum_{k=1}^{\infty} \int_0^Z |\eta_{\epsilon_k} * (c\zeta_k) - c\zeta_k| dz < \epsilon.$$

This means $c_\epsilon \to c$ in $L^1(0, Z)$ as $\epsilon \to 0$.

Next, we need to show that $\mathrm{Var}_{(0,Z)}(c_\epsilon) \to \mathrm{Var}_{(0,Z)}(c)$.

For $\phi \in C_c^1(0, Z)$, $|\phi| \leq 1$ and $c_\epsilon \to c$ in $L^1(0, Z)$,

$$\begin{aligned} &\int_0^Z c\phi' dx = \lim_{\epsilon \to 0} \int_0^Z c_\epsilon \phi' dx = -\lim_{\epsilon \to 0} \int_0^Z \phi c_\epsilon' dx \\ \leq\ &\liminf_{\epsilon \to 0} \int_0^Z |c_\epsilon'| dx \leq \liminf_{\epsilon \to 0} \mathrm{Var}_{(0,Z)}(c_\epsilon) \end{aligned}$$

which gives

$$\mathrm{Var}_{(0,Z)}(c) \leq \liminf_{\epsilon \to 0} \mathrm{Var}_{(0,Z)}(c_\epsilon). \tag{A.2}$$

Also notice that $\sum_{k=1}^{\infty} \zeta_k' = 0$.

$$\int_0^Z c_\epsilon \phi' dx = \sum_{k=1}^{\infty} \int_0^Z \eta_{\epsilon_k} * (c\zeta_k)\phi' dz = \sum_{k=1}^{\infty} \int_0^Z c\zeta_k (\eta_{\epsilon_k} * \phi)' dz$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \int_0^Z c(\zeta_k(\eta_{\epsilon_k} * \phi))dz - \sum_{k=1}^{\infty} \int_0^Z c\zeta_k'(\eta_{\epsilon_k} * \phi)dz \\
&= \sum_{k=1}^{\infty} \int_0^Z c(\zeta_k(\eta_{\epsilon_k} * \phi))dz - \sum_{k=1}^{\infty} \int_0^Z \phi(\eta_{\epsilon_k} * (c\zeta_k') - c\zeta_k')dz
\end{aligned}$$

Notice that if $z \in (0, Z)$, z belongs to at most three of the sets V_k with $k = 1, \cdots$. The first term becomes

$$\begin{aligned}
& |\int_0^Z c(\zeta_1(\eta_{\epsilon_1} * \phi))dz + \sum_{k=2}^{\infty} \int_0^Z c(\zeta_k(\eta_{\epsilon_k} * \phi))dz| \\
\leq \; & \mathrm{Var}_{(0,Z)}(c) + \sum_{k=1}^{\infty} \mathrm{Var}_{V_k}(c) \\
\leq \; & \mathrm{Var}_{(0,Z)}(c) + 3\mathrm{Var}_{(0,Z)-U_1}(c) \\
\leq \; & \mathrm{Var}_{(0,Z)}(c) + 3\epsilon
\end{aligned}$$

By equation (A.1),

$$|\sum_{k=1}^{\infty} \int_0^Z \phi(\eta_{\epsilon_k} * (c\zeta_k') - c\zeta_k')dz| < \epsilon$$

Therefore

$$\int_0^Z c_\epsilon \phi' dx \leq \mathrm{Var}_{(0,Z)}(c) + 4\epsilon,$$

and thus

$$\mathrm{Var}_{(0,Z)}(c_\epsilon) \leq \mathrm{Var}_{(0,Z)}(c) + 4\epsilon.$$

This equation and equation (A.2) give the result.

A.3 Weak solutions to wave equations on domain with a single interface

If c has only one jump, i.e. the material has a single interface (see Figure 2.1), the solution to the wave equation has the form

$$u(z,t) = \begin{cases} U_1(z + c_1 t) + D_1(z - c_1 t), & z \leq z_1 \\ U_2(z + c_2 t) + D_2(z - c_2 t), & z > z_1 \end{cases}$$

where the U_i means the up-coming wave on the interval with velocity c_i and D_i means the down-going wave on the interval with velocity c_i.

Assume the initial conditions are supported at some interval belonging to $(-\infty, z_1]$ (see Figure 2.1). Then $D_1 = 0$, i.e. no down-going wave on the left of z_1. u is a weak solution to the wave equation (1.1).

For test functions $\phi(z,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ and equation (1.1), we have

$$\begin{aligned} 0 &= \iint dtdz \left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}u - \frac{\partial^2\phi}{\partial z^2}u\right) \\ &= \iint dtdz \left(\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\phi - \frac{\partial^2 u}{\partial z^2}\phi\right) - \int dt\frac{\partial\phi}{\partial z}[u] + \int dt\phi\left[\frac{\partial u}{\partial z}\right] \\ &= -\int dt\frac{\partial\phi}{\partial z}[u] + \int dt\phi\left[\frac{\partial u}{\partial z}\right] \end{aligned}$$

Define

$$\eta(z) = \begin{cases} e^{\frac{1}{|z|^2-1}}, & |z| < 1, \\ 0, & |z| > 1. \end{cases}$$

For any $f, g \in C_0^\infty(\mathbb{R})$, define

$$\phi(z,t) = f(t) + g(t)(z - z_1)\eta(z - z_1).$$

It is easy to see that $\phi(z,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ and $\phi(z_1,t) = f(t)$ and $\dfrac{\partial\phi}{\partial z}(z_1,t) = g(t)$.

Thus the above equality gives that $[u] = \lim\limits_{z\to z_1^+} u - \lim\limits_{z\to z_1^-} u = 0$ and $\left[\dfrac{\partial u}{\partial z}\right] = \lim\limits_{z\to z_1^+}\dfrac{\partial u}{\partial z} - \lim\limits_{z\to z_1^-}\dfrac{\partial u}{\partial z} = 0$. This gives us

$$U_1(z_1 + c_1 t) = U_2(z_1 + c_2 t) + D_2(z_1 - c_2 t),$$

$$U_1'(z_1 + c_1 t) = U_2'(z_1 + c_2 t) + D_2'(z_1 - c_2 t).$$

Together with the wave equation and the ray-tracing backwards, this gives us the solution to the wave equation (1.1) with single interface as shown in Figure 2.1. If $t > 0$,

$$u(z,t) = \begin{cases} \dfrac{2c_1}{c_1+c_2}U_2\left(\dfrac{c_2}{c_1}z + \left(1 - \dfrac{c_2}{c_1}\right)z_1 + c_2 t\right) & z \le z_1; \\ \dfrac{c_1-c_2}{c_1+c_2}U_2(2z_1 - z + c_2 t) + U_2(z + c_2 t) + D_2(z - c_2 t) & z \ge z_1. \end{cases}$$

Similarly, if $t < 0$,

$$u(z,t) = \begin{cases} \dfrac{2c_1}{c_1+c_2}D_2\left(\dfrac{c_2}{c_1}z + \left(1 - \dfrac{c_2}{c_1}\right)z_1 - c_2 t\right) & z \le z_1; \\ \dfrac{c_1-c_2}{c_1+c_2}D_2(2z_1 - z - c_2 t) + D_2(z - c_2 t) + U_2(z + c_2 t) & z \ge z_1. \end{cases}$$

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